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On a New Class of Realization Formulas and Their Application

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ABSTRACT

A new set of realization formulas is derived for a class of matrix-valued functions $W(\lambda)$. These include the standard realization formulas for rational $W(\lambda)$. Observability, controllability, and minimality are defined and characterized. Conditions for $W(\lambda)$ to be (J_1, J_2) isometric and/or (J_1, J_2) coisometric with respect to a pair of signature matrices J_1 and J_2 are given in terms of the realizations. Minimal factorizations for square $W(\lambda)$ are considered, and formulas for the factors are deduced. Associated reproducing kernel spaces (of pairs too) are discussed.

1. INTRODUCTION

The purpose of this paper is to derive and develop the theory of a new set of realization formulas for a class of matrix-valued meromorphic functions $W(\lambda)$, which includes the class of rational matrix-valued functions. These realization formulas are of the form

$$W(\lambda) = D + \{a(\alpha)b(\lambda) - b(\alpha)a(\lambda)\}VG(\lambda)^{-1}U,$$

where $a(\lambda)$ and $b(\lambda)$ are suitably restricted analytic functions, which are defined in a connected nonempty open subset Ω of the complex plane,

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$\alpha \in \Omega$, and

$$G(\lambda) = a(\lambda)A - b(\lambda)B$$

is an $n \times n$ matrix-valued function which is analytic in Ω and invertible at the point α. Herein, $D \in \mathbb{C}^{p\times q}$, $V \in \mathbb{C}^{p\times n}$, $A, B \in \mathbb{C}^{n\times n}$, and $U \in \mathbb{C}^{n\times q}$ are all constant matrices; i.e., they are independent of the variable λ. They do, however, depend upon the choice of the point α. These formulas incorporate all the realization formulas in the literature that are known to us as special cases; in particular they include the classical formulas $W(\lambda) = D+\lambda C(I-\lambda A)^{-1}B$ and $W(\lambda) = D+C(\lambda I-A)^{-1}B$, more general formulations in which the identity matrix in the preceding two examples is replaced by a general square matrix to obtain the so-called descriptor form, as well as the formulation discussed by Gohberg, Kaashoek, and Ran [13]. For more on descriptor forms, see, e.g., the papers of Cobb [9], Luenberger [18], Rosenbrock [22], Verghese, Lévy, and Kailath [23], Verghese, Van Dooren, and Kailath [24], Yip and Sinconvec [25], and Zhou, Shayman, and Tarn [26].

Our interest in realization formulas of the stated kind for $W(\lambda)$ originated in our earlier studies of reproducing kernel spaces of a special sort [5–7]. To explain this connection, as well as to proceed further in the present paper, we need some notation.

Throughout this paper Ω denotes an open, nonempty connected subset of the complex plane $\mathbb{C}$. We say that

$$\rho_\omega(\lambda) = a(\lambda)a(\omega)^* - b(\lambda)b(\omega)^* \tag{1.1}$$

belongs to $\mathcal{D}_\Omega$ if a and b are analytic in Ω and both

$$\Omega_+ = \{\omega \in \Omega : \rho_\omega(\omega) > 0\}$$

and

$$\Omega_- = \{\omega \in \Omega : \rho_\omega(\omega) < 0\}$$

are nonempty. Because of the presumed connectedness of Ω, it follows that

$$\Omega_0 = \{\omega \in \Omega : \rho_\omega(\omega) = 0\}$$

is nonempty and furthermore that there exists a point $\mu \in \Omega_0$ such that

$$|a(\mu)| = |b(\mu)| \neq 0;$$

see, e.g., [6] for more details, if needed. In fact more is true: If

$$B_\mu = \{\lambda \in \mathbb{C} : |\mu - \lambda| < \delta\}$$

is a disc about μ of sufficiently small radius δ, then $B_\mu \subset \Omega$, $a(\lambda) \neq 0$ in B_μ and $s(\lambda) = b(\lambda)/a(\lambda)$ is a nonconstant analytic function in B_μ. Therefore, by the open mapping theorem for analytic functions, $s(B_\mu)$ is an open connected set. Since $|s(\mu)| = 1$, it contains an open subarc of the unit circle $\mathbb{T}$. This proves that

$$\left\{ \frac{b(\lambda)}{a(\lambda)} : \lambda \in \Omega_0 \text{ and } a(\lambda) \neq 0 \right\} \supset \text{ an open subarc of } \mathbb{T}. \tag{1.2}$$

In particular (1.2) guarantees that if f is analytic in Ω and $f(\lambda) = 0$ for every point $\lambda \in \Omega_0$, then $f(\lambda) \equiv 0$.

We first encountered functions $\rho_\omega(\lambda)$ of the special form (1.1) in the work of Lev-Ari and Kailath [16] who used them to unify their study of efficient algorithms for the triangular factorization of a class of structured matrices. Several years later (at the Workshop on Operator Theory and its Applications in Sapporo, 1991) we learned that such functions were introduced independently by A. A. Nudelman and some of his associates to formulate a unified approach to a number of interpolation problems that was close in spirit to the then already submitted manuscript [6]. For a recent account of this work, see [17].

A number of examples of the classes $\mathcal{D}_\Omega$ are furnished in [6]. In particular it is also shown there that the choice of the functions $a(\lambda)$ and $b(\lambda)$ in (1.1) is essentially unique: If

$$\rho_\omega(\lambda) = c(\lambda)c(\omega)^* - d(\lambda)d(\omega)^*, \qquad (\lambda, \omega \in \Omega) \tag{1.3}$$

for a second pair of functions c and d, which are analytic in Ω, then there is a constant 2×2 matrix M which is unitary with respect to the signature matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

such that

$$[c(\lambda) \quad d(\lambda)] = [a(\lambda) \quad b(\lambda)]M. \tag{1.4}$$

In the sequel we shall make extensive use of the function

$$\delta_\omega(\lambda) = a(\omega)b(\lambda) - b(\omega)a(\lambda). \tag{1.5}$$

This too depends essentially upon ρ and not upon the choice of a and b in the representation (1.1). More precisely, if (1.3) also holds for a second pair of functions c and d, which are analytic in Ω, then by (1.4),

$$\begin{bmatrix} c(\omega) & d(\omega) \\ c(\lambda) & d(\lambda) \end{bmatrix} = \begin{bmatrix} a(\omega) & b(\omega) \\ a(\lambda) & b(\lambda) \end{bmatrix} M$$

and hence,

$$c(\omega)d(\lambda) - d(\omega)c(\lambda) = \{a(\omega)b(\lambda) - b(\omega)a(\lambda)\} \det M. \tag{1.6}$$

(Since M is unitary with respect to the indicated signature matrix, $|\det M| = 1$.)

Formula (1.4) also indicates that $a(\omega) = b(\omega) = 0$ if and only if $c(\omega) = d(\omega) = 0$. Thus this "property" is also independent of the decomposition of $\rho_\omega(\lambda)$. This is clarified further by the following observation:

LEMMA 1.1. *The following three conditions are equivalent for every point $\omega \in \Omega$:*

(1) $\delta_\omega(\lambda) \equiv 0$ *(as a function of λ).*
(2) $\rho_\omega(\lambda) \equiv 0$ *(as a function of λ).*
(3) $a(\omega) = b(\omega) = 0$.

Proof. The implications (3) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are self-evident. On the other hand, if either $a(\omega) \neq 0$ or $b(\omega) \neq 0$, then both (1) and (2) must fail because otherwise $a(\lambda)$ and $b(\lambda)$ would be linearly dependent functions of λ, which contradicts the assumption that Ω_+ and Ω_- are both nonempty. ∎

Finite-dimensional reproducing kernel spaces of the type alluded to above can always be expressed in the form

$$\mathcal{M} = \left\{ VG(\lambda)^{-1}x : x \in \mathbb{C}^{n\times n} \right\},$$

where the columns of $VG(\lambda)^{-1}$ are linearly independent in the vector space of continuous $p \times 1$ vector-valued functions on Ω. These spaces are endowed with an inner product (possible indefinite) that is based on an invertible Hermitian matrix P by the rule

$$[VG(\lambda)^{-1}x, VG(\lambda)^{-1}y]_{\mathcal{M}} = y^*Px.$$

Such a space $\mathcal{M}$ is an n-dimensional reproducing kernel Krein space with a reproducing kernel $K_\omega(\lambda)$, which is given by the formula

$$K_\omega(\lambda) = VG(\lambda)^{-1}P^{-1}G(\omega)^{-*}V^*$$

for every pair of points λ and ω in Ω at which the indicated inverses exist. In particular, this means that

(1) $K_\omega x \in \mathcal{M}$, and
(2) $[f, K_\omega x]_{\mathcal{M}} = x^* f(\omega)$

for every point $\omega \in \Omega$ at which $G(\omega)$ is invertible, every vector $x \in \mathbb{C}^n$, and every $f \in \mathcal{M}$.

Under additional constraints on P, the kernel can also be expressed in the form

$$K_\omega(\lambda) = \frac{J - W(\lambda)JW(\omega)^*}{\rho_\omega(\lambda)},$$

where J is a signature matrix and $W(\lambda)$ is a $p \times p$ matrix-valued function that is meromorphic in Ω. Upon matching the two reproducing kernel formulas, it is readily seen that

$$W(\lambda)JW(\omega)^* = \{I_p - \rho_\omega(\lambda)VG(\lambda)^{-1}P^{-1}G(\omega)^{-*}V^*J\}J.$$

Thus, if β is a point in Ω_0 at which $G(\beta)$ is invertible, we obtain first that

$$W(\beta)JW(\beta)^* = J$$

and hence that

$$W(\lambda) = \left\{I_p - \rho_\beta(\lambda)VG(\lambda)^{-1}P^{-1}G(\beta)^{-*}V^*J\right\} W(\beta).$$

This is a realization formula of the form exhibited above because for such points β, $|a(\beta)| = |b(\beta)| \neq 0$ and

$$a(\beta)\rho_\beta(\lambda) = -\delta_\beta(\lambda)b(\beta)^*,$$

i.e.,

$$D = W(\beta) \qquad \text{and} \qquad U = \frac{b(\beta)^*}{a(\beta)}P^{-1}G(\beta)^{-*}V^*JW(\beta).$$

A more general derivation via reproducing kernels of realization formulas of this form for $p \times p$ matrix-valued functions $W(\lambda)$, which are not necessarily J unitary on Ω_0, is furnished in Section 4. We turn now, however, to a different method that is applicable to nonsquare matrix-valued functions also. For every $p \times q$ matrix-valued function W that is meromorphic in Ω, let

$$\mathcal{R}_c(W) = \left\{\text{finite sums of the form} \sum_{j=1}^{k} \frac{W(\lambda) - W(\omega_j)}{\delta_{\omega_j}(\lambda)}\eta_j\right\} \tag{1.7}$$

and

$$\mathcal{R}_{\mathrm{r}}(W) = \left\{ \text{finite sums of the form} \sum_{j=1}^{k} \xi_j^* \frac{W(\lambda) - W(\omega_j)}{\delta_{\omega_j}(\lambda)} \right\}, \tag{1.8}$$

wherein the ξ_j run over $\mathbb{C}^p$, the η_j run over $\mathbb{C}^q$, and the ω_j run through the set of points in Ω at which W is analytic and $|a(\omega_j)| + |b(\omega_j)| \neq 0$ (and hence, by Lemma 1.1, $\delta_{\omega_j}(\lambda) \not\equiv 0$ as a function of λ).

In the sequel (Theorem 2.2) we shall show that

$$\dim \mathcal{R}_{\mathrm{c}}(W) = \dim \mathcal{R}_{\mathrm{r}}(W).$$

When this number is finite we shall refer to it as the degree of W with respect to ρ (and denote it by $\deg_\rho W$); the fact that it depends upon ρ and not upon the particular choice of a and b in (1.1) is immediate from (1.6). For the three classical choices of ρ; $\rho_\omega(\lambda) = 1 - \lambda\omega^*$, $\rho_\omega(\lambda) = -2\pi i(\lambda - \omega^*)$, and $\rho_\omega(\lambda) = 2\pi(\lambda + \omega^*)$, with Ω_+ equal to the open unit disc $\mathbb{D}$, the open upper halfplane $\mathbb{C}_+$, and the open right halfplane Π_+, respectively, this turns out to be just the McMillan degree of W; see [2].

The paper is organized as follows. In Section 2 we establish the realization formulas described above for a general class of $p \times q$ matrix-valued functions $W(\lambda)$ that are meromorphic in Ω. We also derive a number of auxiliary formulas and results for later use. Next, in Section 3, we discuss the concepts of observability, controllability, and minimality, which are appropriate for the present class of realization formulas, and give a number of equivalent characterizations of each. The similarity of two minimal realizations is also discussed there. In Section 4 we develop necessary and sufficient conditions in terms of Lyapunov–Stein-type equations for the existence of realizations of this general type, which are either isometric or coisometric (or both) matrix-valued functions with respect to a pair of (or just one) signature matrices on the boundary of an appropriately defined region. In Section 5 we discuss pairings wherein a given matrix-valued function $W(\lambda)$ is "paired" with a partner $W_\pi(\lambda)$, which is often an effective substitute for $W(\lambda)^*$ when the conditions for $W(\lambda)$ to be either isometric or coisometric (with respect to two given signature matrices) on the boundary fail. Section 6 is devoted to embedding the basic Lyapunov–Stein-type equations into a larger array of equations and related implications. Section 7 treats factorization and minimal factorization.

We shall use the symbol Ω_W to denote the domain of analyticity of W in Ω, for any matrix-valued function W that is defined in Ω.

The other notation is mostly standard: $\mathbb{C}$, $\mathbb{R}$, $\mathbb{T}$ denote the complex numbers, the reals, and the unit circle, respectively; $\mathbb{C}^{m\times n}$ is the set of $m \times n$ matrices with complex entries; and $\mathbb{C}^m$ is short for $\mathbb{C}^{m\times 1}$. The superscript $*$

applied to a matrix A as in A^* denotes the Hermitian transpose of A (i.e., the adjoint with respect to the standard inner product in complex Euclidean space), whereas A^τ denotes the ordinary transpose. If A is a number, then A^* is just its complex conjugate.

A signature matrix A is a matrix that is both Hermitian and unitary: $A = A^*$ and $A^*A = I$. If A is Hermitian, then $\mu_+(A)$ [resp. $\mu_-(A)$] denotes the number of positive [resp. negative] eigenvalues of A.

2. THE SPACES $\mathcal{R}_c(W)$ AND $\mathcal{R}_r(W)$

In this section we develop a general realization formula for matrix-valued meromorphic functions W for which the associated spaces $\mathcal{R}_c(W)$ and $\mathcal{R}_r(W)$ are finite dimensional. To this end, the operators

$$\begin{aligned}\{r(a,b;\alpha)f\}(\lambda) &= \frac{a(\lambda)f(\lambda)-a(\alpha)f(\alpha)}{\delta_\alpha(\lambda)}\\ \{r(b,a;\alpha)f\}(\lambda) &= -\frac{b(\lambda)f(\lambda)-b(\alpha)f(\alpha)}{\delta_\alpha(\lambda)},\end{aligned}$$

which were introduced in [7], prove useful.

2.1. Realizations, First Approach

LEMMA 2.1. *Let W be a $p\times q$ matrix-valued function that is meromorphic in Ω, let*

$$\Phi_\omega(\lambda) = \frac{W(\lambda)-W(\omega)}{\delta_\omega(\lambda)} \tag{2.1}$$

for short, and let

$$f=\sum_{j=1}^{k}\Phi_{\omega_j}\eta_j \qquad \left[\text{resp.}\quad f=\sum_{j=1}^{k}\xi_j^*\Phi_{\omega_j}\right]$$

be an element of $\mathcal{R}_c(W)$ [resp. $\mathcal{R}_r(W)$] for some choice of points $\omega_j\in\Omega_W$ at which $|a(\omega_j)|+|b(\omega_j)|>0$, $j=1,\dots,k$. Then both $r(a,b;\alpha)f$ and $r(b,a;\alpha)f$ belong to $\mathcal{R}_c(W)$ [resp. $\mathcal{R}_r(W)$] for every $\alpha\in\Omega_W$ at which $\delta_{\omega_j}(\alpha)\neq 0$, $j=1,\dots,k$.

Proof. With the help of the pair of identities

$$a(\omega)\delta_\alpha(\lambda)-a(\lambda)\delta_\alpha(\omega)=a(\alpha)\delta_\omega(\lambda) \tag{2.2}$$

and

$$b(\omega)\delta_\alpha(\lambda) - b(\lambda)\delta_\alpha(\omega) = b(\alpha)\delta_\omega(\lambda), \tag{2.3}$$

it is readily checked that

$$\{r(a,b;\alpha)\Phi_\omega\}(\lambda) = \frac{a(\alpha)\Phi_\alpha(\lambda) - a(\omega)\Phi_\omega(\lambda)}{\delta_\omega(\alpha)} \tag{2.4}$$

and

$$\{r(b,a;\alpha)\Phi_\omega\}(\lambda) = \frac{b(\omega)\Phi_\omega(\lambda) - b(\alpha)\Phi_\alpha(\lambda)}{\delta_\omega(\alpha)} \tag{2.5}$$

when $\delta_\alpha(\omega) \neq 0$. The stated result now emerges easily upon combining formulas. ■

The next result is adapted from the analysis in [10, Sect. 3.1].

LEMMA 2.2. *Let $f_1, \ldots, f_n$ be a basis for an n-dimensional vector space $\mathcal{M}$ of $m \times 1$ vector-valued functions that are meromorphic in Ω, and suppose that either $\mathcal{M}$ is $r(a,b;\alpha)$ invariant for a point $\alpha \in \Omega$ in the common domain of analyticity of $f_1, \ldots, f_n$ at which $a(\alpha) \neq 0$ or $\mathcal{M}$ is $r(b,a;\alpha)$ invariant for some point $\alpha \in \Omega$ in the common domain of analyticity of $f_1, \ldots, f_n$ at which $b(\alpha) \neq 0$. Then the $m \times n$ matrix-valued meromorphic function*

$$F(\lambda) = [f_1(\lambda) \cdots f_n(\lambda)]$$

can be expressed in the form

$$F(\lambda) = V\{a(\lambda)A - b(\lambda)B\}^{-1} \tag{2.6}$$

for some choice of $V \in \mathbb{C}^{m\times n}$, $A \in \mathbb{C}^{n\times n}$, and $B \in \mathbb{C}^{n\times n}$, with $AB = BA$ and

$$\det\{a(\alpha)A - b(\alpha)B\} \neq 0.$$

Moreover, the matrix

$$G(\lambda) = a(\lambda)A - b(\lambda)B$$

is invertible if and only if $\lambda \in \Omega_F$, the domain of analyticity of F in Ω.

Proof. Under the first assumption, there exists a choice of constants $c_{ij}(\alpha)$, $i, j = 1, \dots, n$, such that

$$r(a, b; \alpha) f_j = \sum_{i=1}^{n} f_i c_{ij}(\alpha),$$

for $j = 1, \dots, n$. But this is the same as to say that there exists an $n \times n$ matrix C_α such that

$$\frac{a(\lambda)F(\lambda) - a(\alpha)F(\alpha)}{\delta_\alpha(\lambda)} = F(\lambda)C_\alpha.$$

(Lemma 1.1 guarantees that $\delta_\alpha(\lambda) \not\equiv 0$.) Therefore,

$$F(\lambda)\{a(\lambda)I_n - \delta_\alpha(\lambda)C_\alpha\} = a(\alpha)F(\alpha),$$

which can clearly be expressed in the indicated form with

$$A = I_n + b(\alpha)C_\alpha, \qquad B = a(\alpha)C_\alpha, \qquad V = a(\alpha)F(\alpha).$$

Moreover,

$$\det\{a(\alpha)A - b(\alpha)B\} = \det\{a(\alpha)I_n\} \neq 0,$$

since $a(\alpha) \neq 0$ by assumption, and $AB = BA$.

On the other hand, if $\mathcal{M}$ is $r(b, a; \alpha)$ invariant for some point $\alpha \in \Omega$ in the domain of analyticity of F for which $b(\alpha) \neq 0$, then the same conclusion holds by much the same argument, but with a and b interchanged.

Finally, since a point ω clearly belongs to Ω_F if $G(\omega)$ is invertible, it remains only to check that the converse is true also. The argument is split into two steps.

Step 1. If $\omega \in \Omega_F$, then $\delta_\omega(\lambda) \not\equiv 0$.

Proof of Step 1. If $\delta_\omega(\lambda) \equiv 0$, then by Lemma 1.1, $a(\omega) = b(\omega) = 0$. Therefore

$$F(\omega)G(\omega) = V = 0,$$

which implies in turn that $F(\lambda)G(\lambda) \equiv 0$ and hence, since G is invertible in a neighborhood of α, that $F(\lambda) \equiv 0$. This contradicts the presumed linear independence of the columns of F.

Step 2. If $\omega \in \Omega_F$, then $G(\omega)$ is invertible.

Proof of Step 2. If $G(\omega)u = 0$ for some $u \in \mathbb{C}^n$, then

$$F(\lambda)G(\lambda)u = Vu = 0,$$

first by evaluating the left-hand side at ω and then for all λ since V is independent of λ. Thus

$$\delta_\omega(\lambda)F(\lambda)Bu = F(\lambda)\{a(\lambda)G(\omega) - a(\omega)G(\lambda)\}u = 0$$

and

$$\delta_\omega(\lambda)F(\lambda)Au = F(\lambda)\{b(\lambda)G(\omega) - b(\omega)G(\lambda)\}u = 0,$$

for every $\lambda \in \Omega_F$. Thus by Step 1 and the presumed linear independence of the columns of F,

$$Au = Bu = 0,$$

which implies that

$$G(\alpha)u = 0$$

and hence that $u = 0$. Therefore $G(\omega)$ is invertible, and the proof of both the step and the lemma is complete. ∎

LEMMA 2.3. *Let $f_1, \dots, f_n$ be a basis for an n-dimensional vector space $\mathcal{M}$ of $1 \times m$ vector-valued functions that are meromorphic in Ω, and suppose that either $\mathcal{M}$ is $r(a,b;\alpha)$ invariant for a point $\alpha \in \Omega$ in the common domain of analyticity of $f_1, \dots, f_n$ at which $a(\alpha) \neq 0$ or $\mathcal{M}$ is $r(b,a;\alpha)$ invariant for some point $\alpha \in \Omega$ in the common domain of analyticity of $f_1, \dots, f_n$ at which $b(\alpha) \neq 0$. Then the $n \times m$ matrix-valued meromorphic function*

$$F(\lambda) = \begin{bmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{bmatrix}$$

can be expressed in the form

$$F(\lambda) = \{a(\lambda)A - b(\lambda)B\}^{-1}U \tag{2.7}$$

for some choice of $U \in \mathbb{C}^{n\times m}$, $A \in \mathbb{C}^{n\times n}$, and $B \in \mathbb{C}^{n\times n}$, with $AB = BA$ and

$$\det\{a(\alpha)A - b(\alpha)B\} \neq 0.$$

Moreover, the matrix

$$G(\lambda) = a(\lambda)A - b(\lambda)B$$

is invertible if and only if $\lambda \in \Omega_F$, *the domain of analyticity of* F *in* Ω.

Proof. The proof is much the same as the proof of Lemma 2.2. ∎

THEOREM 2.1. *Let* $\rho \in \mathcal{D}_\Omega$ *and let* W *be a* $p \times q$ *matrix-valued function that is meromorphic in* Ω *such that either* $\mathcal{R}_c(W)$ *or* $\mathcal{R}_r(W)$ *is a finite-dimensional vector space of dimension* n. *Let* $\alpha \in \Omega_W$ *be such that* $\rho_\alpha(\lambda) \not\equiv 0$. *Then* W *admits a representation of the form*

$$W(\lambda) = D + \delta_\alpha(\lambda)V\{a(\lambda)A - b(\lambda)B\}^{-1}U, \tag{2.8}$$

where $D \in \mathbb{C}^{p\times q}, V \in \mathbb{C}^{p\times n}, U \in \mathbb{C}^{n\times q}$, $A, B \in \mathbb{C}^{n\times n}$, $AB = BA$, *and*

$$G(\lambda) = a(\lambda)A - b(\lambda)B \tag{2.9}$$

is invertible at the point α. *In particular,* $D = W(\alpha)$ *and* $W(\lambda)$ *is a rational function of* $s(\lambda) = b(\lambda)/a(\lambda)$.

Proof. Suppose first that $\mathcal{R}_c(W)$ is a finite-dimensional space with basis $f_1, \ldots, f_n$. Then by Lemma 2.1, $\mathcal{R}_c(W)$ is $r(a, b; \alpha)$ invariant for some $\alpha \in \Omega$ in the common domain of analyticity of W and $f_1, \ldots, f_n$ with $a(\alpha) \neq 0$. Therefore by Lemma 2.2, the $p \times n$ matrix-valued function $F = [f_1 \cdots f_n]$ can be expressed in the form

$$F(\lambda) = V\{a(\lambda)A - b(\lambda)B\}^{-1},$$

where the sizes of V, A, and B are as in the statement of the theorem and $G(\alpha)$ is invertible. Thus there exists an $n \times q$ matrix $U = U_\alpha$ such that

$$\frac{W(\lambda) - W(\alpha)}{\delta_\alpha(\lambda)} = V\{a(\lambda)A - b(\lambda)B\}^{-1}U, \tag{2.10}$$

which is the same as the stated representation.

This completes the proof of the theorem when $\mathcal{R}_c(W)$ is finite dimensional. The proof under the assumption that $\mathcal{R}_r(W)$ is finite dimensional goes through in much the same way. ∎

We remark that the term $a(\lambda)A - b(\lambda)B$ in (2.8) is not necessarily a matrix polynomial of degree one in the variable λ. Thus for example, if $a(\lambda)$, $b(\lambda)$, A, and B are chosen as in [6, Sect. 6] (which studies block matrices that are of the form block Hankel plus block Toeplitz), then the realization formula (2.8) is of the form

$$W(\lambda) = D - (\lambda - \alpha)(1 - \lambda\alpha)V\{(\lambda I - Z)(I - \lambda Z)\}^{-1}U,$$

where Z is an upper triangular block (downward) shift matrix. More generally, $G(\lambda)$ will be an analytic function in Ω; see [6, 17] for a number of choices of $a(\lambda)$ and $b(\lambda)$.

The realization exhibited in (2.8) is minimal in the sense that A and B are $n \times n$ and $n = \dim \mathcal{R}_c(W) = \dim \mathcal{R}_r(W)$ (see also Theorem 2.2, below).

2.2. *Auxiliary Formulas and Facts*

The following elementary identity will prove useful in future applications of the preceding realization formula.

LEMMA 2.4. *If $G(\alpha) = a(\alpha)A - b(\alpha)B$ is invertible, then*

$$AG(\alpha)^{-1}B = BG(\alpha)^{-1}A. \tag{2.11}$$

Proof. Clearly

$$\begin{aligned} a(\alpha)AG(\alpha)^{-1}B &= \{G(\alpha) + b(\alpha)B\}G(\alpha)^{-1}B \\ &= B + BG(\alpha)^{-1}b(\alpha)B \\ &= B + BG(\alpha)^{-1}\{a(\alpha)A - G(\alpha)\} \\ &= a(\alpha)BG(\alpha)^{-1}A. \end{aligned}$$

This serves to prove (2.11) if $a(\alpha) \neq 0$. If $a(\alpha) = 0$, then $b(\alpha) \neq 0$ and much the same argument goes through once again. ■

From now on we refer to formula (2.8) with D, V, U, A, and B of the sizes indicated in the statement of Theorem 2.1, and $G(\alpha)$ invertible, as a 'ρ' realization of W centered at α. The terminology is justified because the form of the representation depends only upon ρ and not upon the choice of $a(\lambda)$ and $b(\lambda)$. Thus if $\rho_\omega(\lambda)$ is expressed in terms of $c(\lambda)$ and $d(\lambda)$ as in (1.3), then by (1.4),

$$\begin{aligned} G(\lambda) &= [a(\lambda)I_n \quad b(\lambda)I_n]\begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} \\ &= [c(\lambda)I_n \quad d(\lambda)I_n]\begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}, \end{aligned}$$

with

$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} m_{11}^* I_n & m_{21}^* I_n \\ m_{12}^* I_n & m_{22}^* I_n \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}.$$

Consequently, by (1.6),

$$W(\lambda) = D + \{c(\alpha)d(\lambda) - d(\alpha)c(\lambda)\}\{\det M^*\}V\{c(\lambda)A_1 - d(\lambda)B_1\}^{-1}U,$$

which is the same form as (2.8). Moreover, it is readily checked that

$$A_1B_1 - B_1A_1 = (AB - BA)\det M^*.$$

LEMMA 2.5. *Let W be a $p \times q$ matrix-valued meromorphic function in Ω that admits a realization of the form*

$$W(\lambda) = D + \delta_\alpha(\lambda)V\{a(\lambda)A - b(\lambda)B\}^{-1}U, \tag{2.12}$$

where $\alpha \in \Omega_W$, A, and B are $n \times n$ matrices, and λ is a point in Ω_W at which the indicated inverse exists. Suppose further that ω is a point in Ω_W such that $G(\omega)$ is invertible. Then

$$\begin{aligned} \frac{W(\lambda) - W(\omega)}{\delta_\omega(\lambda)} &= VG(\lambda)^{-1}G(\alpha)G(\omega)^{-1}U \\ &= VG(\omega)^{-1}G(\alpha)G(\lambda)^{-1}U. \end{aligned} \tag{2.13}$$

Proof. The verification of (2.13) is a straightforward calculation, with the help of the identities (2.2) and (2.3). ■

THEOREM 2.2. *If W is a $p \times q$ matrix-valued function that is meromorphic in Ω, then the spaces $\mathcal{R}_c(W)$ and $\mathcal{R}_r(W)$ are either both infinite dimensional or they are both finite dimensional with the same dimension. In either case,*

$$\dim \mathcal{R}_c(W) = \dim \mathcal{R}_r(W). \tag{2.14}$$

Proof. Either $\mathcal{R}_c(W)$ and $\mathcal{R}_r(W)$ are both infinite dimensional, or at least one of these spaces is finite dimensional. If say $\dim \mathcal{R}_c(W) = n < \infty$, then by Theorem 2.1, W admits a realization of the form (2.12) with a $G(\lambda)$ of size $n \times n$. Thus by (2.13),

$$\sum \xi_j^* \Phi_{\omega_j} = \sum \xi_j^* VG(\omega_j)^{-1}G(\alpha)G(\lambda)^{-1}U, \tag{2.15}$$

which serves to exhibit every finite sum of the form indicated on the left of the last formula as a finite linear combination of the n rows of $G(\lambda)^{-1}U$. Therefore

$$\dim \mathcal{R}_{\rm r}(W) \leq n = \dim \mathcal{R}_{\rm c}(W).$$

On the other hand, if dim $\mathcal{R}_{\rm r}(W) = k < \infty$, then by another application of Theorem 2.1 and (2.13),

$$\sum \Phi_{\omega_j}\eta_j = \sum VG(\lambda)^{-1}G(\alpha)G(\omega_j)^{-1}U\eta_j, \tag{2.16}$$

except that now G is $k \times k$. This exhibits the sum on the left as a linear combination of the k columns of $VG(\lambda)^{-1}$. Thus

$$n = \dim \mathcal{R}_{\rm c}(W) \leq \dim \mathcal{R}_{\rm r}(W) = k.$$

Therefore equality prevails. ■

COROLLARY. *If $W(\lambda)$ is a $p \times q$ matrix-valued function that is meromorphic in Ω and admits a realization of the form* (2.8) *with A and B of size $n \times n$, then:*

(1) $\mathcal{R}_{\rm c}(W) \subseteq \{VG(\lambda)^{-1}x : x \in \mathbb{C}^n\}$; *equality holds if and only if*

$$\text{linear span}\{G(\omega)^{-1}U\eta : \omega \in \Omega_{G^{-1}} \text{ and } \eta \in \mathbb{C}^q\} = \mathbb{C}^n$$

(*i.e., in terms of future terminology, if and only if the triple $\{A, B, U\}$ is (a, b) controllable*).

(2) $\mathcal{R}_{\rm r}(W) \subseteq \{y^*G(\lambda)^{-1}U : y \in \mathbb{C}^n\}$; *equality holds if and only if*

$$\text{linear span}\{\xi^*VG(\omega)^{-1} \ : \ \omega \in \Omega_{G^{-1}} \text{and } \xi \in \mathbb{C}^p\} = \mathbb{C}^{1\times n}$$

(*i.e., in terms of future terminology, if and only if the triple $\{V, A, B\}$ is (a, b) observable*).

Proof. This is an easy consequence of (2.13). The parenthetic remarks depend upon Lemma 3.2. ■

THEOREM 2.3. *If $\mathcal{R}_{\rm c}(W)$ and $R_{\rm r}(W)$ are finite dimensional, then in definitions* (1.7) *and* (1.8), *it suffices to let ω run over any infinite subset Δ of Ω_W that contains at least one limit point and in which $|a(\omega)| + |b(\omega)| > 0$ (see Lemma* 1.1 *for more information on the implications of this assumption).*

Proof. Let

$$\mathcal{R}_{\mathrm{c}}^{\Delta}(W) = \left\{ \text{finite sums of the form} \quad \sum_{j=1}^{k} \Phi_{\omega_j}\eta_j : \omega_j \in \Delta, \quad \eta_j \in \mathbb{C}^q \right\}$$

for any such set Δ and suppose that

$$\dim \mathcal{R}_{\mathrm{c}}^{\Delta}(W) < \dim \mathcal{R}_{\mathrm{c}}(W) = n < \infty.$$

Suppose further that $\alpha \in \Delta$ is such that $G(\alpha)$ is invertible. Then since $VG(\lambda)^{-1}$ has n linearly independent columns, it follows from (2.13) that

$$\{G(\omega)^{-1}U\eta : \eta \in \mathbb{C}^q, \omega \in \Delta\}$$

is a vector space of dimension smaller than n. Therefore, there exists a vector $x \in \mathbb{C}^n$ such that

$$x^*G(\omega)^{-1}U\eta = 0$$

for every choice of $\omega \in \Delta$ and $\eta \in \mathbb{C}^q$. But since Ω is connected, this propagates to every point $\omega \in \Omega$ at which $G(\omega)$ is invertible and therefore implies that $\dim \mathcal{R}_{\mathrm{c}}(W) < n$, contrary to assumption.

The proof for $\mathcal{R}_{\mathrm{r}}(W)$ is similar. ■

For the classical choices of $\rho_\omega(\lambda)$, (2.12) reduces to the more familiar realization formulas that appear throughout the literature on system theory and control. For a derivation of classical realization formulas by these methods, see, e.g., [10, Sect. 4] which served as a model for the present argument in this more general setting.

LEMMA 2.6. *Let*

$$E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

be a block matrix with invertible diagonal block entries A and D. Then the following are equivalent:

(1) *E is invertible*
(2) *$D - CA^{-1}B$ is invertible*
(3) *$A - BD^{-1}C$ is invertible.*

Moreover, if any one (and hence all three) of these last three conditions are in force, then

$$(D - CA^{-1}B)^{-1} = D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1},$$

and

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}.$$

Proof. The first statement is immediate from the well-known Schur complement formulas. The last two formulas are straightforward computations. ■

2.3. Multiplication Formulas

Let $W_i(\lambda)$, $i = 1, 2$, be a pair of $p \times p$ matrix-valued functions that admit realizations of the form (2.8):

$$W_i(\lambda) = D_i + \delta_\alpha(\lambda)V_iG_i(\lambda)^{-1}U_i,$$

in which the $n \times n$ matrix-valued functions

$$G_i(\lambda) = a(\lambda)A_i - b(\lambda)B_i, \qquad i = 1, 2,$$

are both invertible at the point $\alpha \in \Omega$. Then it is readily checked that the product

$$W(\lambda) = W_1(\lambda)W_2(\lambda)$$

can be expressed in the form

$$W(\lambda) = D_1D_2 + \delta_\alpha(\lambda)[V_1 \quad D_1V_2] \begin{bmatrix} G_1(\lambda) & -\delta_\alpha(\lambda)U_1V_2 \\ 0 & G_2(\lambda) \end{bmatrix}^{-1} \begin{bmatrix} U_1D_2 \\ U_2 \end{bmatrix}.$$

Since

$$\begin{bmatrix} G_1(\lambda) & -\delta_\alpha(\lambda)U_1V_2 \\ 0 & G_2(\lambda) \end{bmatrix} = a(\lambda) \begin{bmatrix} A_1 & b(\alpha)U_1V_2 \\ 0 & A_2 \end{bmatrix} - b(\lambda) \begin{bmatrix} B_1 & a(\alpha)U_1V_2 \\ 0 & B_2 \end{bmatrix},$$

it is now easily seen that the preceding formula for $W(\lambda)$ is a realization of the form (2.8) with

$$A = \begin{bmatrix} A_1 & b(\alpha)U_1V_2 \\ 0 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & a(\alpha)U_1V_2 \\ 0 & B_2 \end{bmatrix}, \qquad U = \begin{bmatrix} U_1D_2 \\ U_2 \end{bmatrix},$$
$$D = D_1D_2 \qquad \text{and} \qquad V = [V_1 \quad D_1V_2].$$

If $a(\lambda) = 1$ and $b(\lambda) = \lambda$, then the last set of formulas reduce to those given in (2.5) of [13].

2.4. More on Realizations

In this subsection we present a generalization of the realization formula (2.8) which is valid even when the space $\mathcal{R}_c(W)$ is not finite dimensional. Accordingly we assume throughout that the hypotheses of Theorem 2.1 are in force except that we do not invoke the assumption that at least one of the spaces $\mathcal{R}_c(W)$ or $\mathcal{R}_r(W)$ is finite dimensional.

The first step is to introduce the operators

$$\mathbf{A}: f \in \mathcal{R}_c(W) \to -r(b,a;\alpha)f \in \mathcal{R}_c(W), \tag{2.17}$$

$$\mathbf{B}: f \in \mathcal{R}_c(W) \to r(a,b;\alpha)f \in \mathcal{R}_c(W), \tag{2.18}$$

$$\mathbf{U}: \eta \in \mathbb{C}^q \to \frac{W(\lambda) - W(\alpha)}{\delta_\alpha(\lambda)}\eta \in \mathcal{R}_c(W), \tag{2.19}$$

$$\mathbf{V}: f \in \mathcal{R}_c(W) \to f(\alpha) \in \mathbb{C}^p, \tag{2.20}$$

$$\mathbf{D}: \eta \in \mathbb{C}^q \to W(\alpha)\eta \in \mathbb{C}^p, \tag{2.21}$$

the first three of which were introduced in [7].

The second step is to verify the formula

$$\{(a(\omega)\mathbf{A} - b(\omega)\mathbf{B})^{-1}f\}(\lambda) = \frac{\delta_\alpha(\lambda)f(\lambda) - \delta_\alpha(\omega)f(\omega)}{\delta_\omega(\lambda)} \tag{2.22}$$

for every point $\omega \in \Omega$ in the domain of analyticity of $f \in \mathcal{R}_c(W)$ at which $\delta_\omega(\lambda) \not\equiv 0$. To this end let

$$h(\lambda) = \{(a(\omega)\mathbf{A} - b(\omega)\mathbf{B})^{-1}f\}(\lambda).$$

Then clearly

$$\begin{aligned} f(\lambda) &= \{(a(\omega)\mathbf{A} - b(\omega)\mathbf{B})h\}(\lambda) \\ &= a(\omega)\frac{b(\lambda)h(\lambda) - b(\alpha)h(\alpha)}{\delta_\alpha(\lambda)} - b(\omega)\frac{a(\lambda)h(\lambda) - a(\alpha)h(\alpha)}{\delta_\alpha(\lambda)} \\ &= \frac{\delta_\omega(\lambda)h(\lambda) - \delta_\omega(\alpha)h(\alpha)}{\delta_\alpha(\lambda)}. \end{aligned}$$

Now, if $\delta_\omega(\alpha) = 0$, then

$$h(\lambda) = \frac{\delta_\alpha(\lambda)}{\delta_\omega(\lambda)}f(\lambda),$$

whereas if $\delta_\omega(\alpha) \neq 0$, then

$$f(\omega) = h(\alpha)$$

and hence

$$h(\lambda) = \frac{\delta_\alpha(\lambda) f(\lambda) - \delta_\alpha(\omega) f(\omega)}{\delta_\omega(\lambda)}.$$

Thus the second formula for $h(\lambda)$ is valid whether $\delta_\alpha(\omega) = 0$ or not and so serves to verify formula (2.22).

The particular choice $f = \mathbf{U}\eta$ leads to

$$(a(\omega)\mathbf{A} - b(\omega)\mathbf{B})^{-1}\mathbf{U}\eta = \frac{W(\lambda) - W(\omega)}{\delta_\omega(\lambda)}\eta,$$

after a short calculation. Therefore,

$$\begin{aligned}\mathbf{V}(a(\omega)\mathbf{A} - b(\omega)\mathbf{B})^{-1}\mathbf{U}\eta &= \frac{W(\alpha) - W(\omega)}{\delta_\omega(\alpha)}\eta \\ &= \frac{W(\omega) - W(\alpha)}{\delta_\alpha(\omega)}\eta,\end{aligned}$$

which in turn supplies the realization

$$W(\omega)\eta = \{\mathbf{D} + \mathbf{V}(a(\omega)\mathbf{A} - b(\omega)\mathbf{B})^{-1}\mathbf{U}\}\eta. \tag{2.23}$$

We remark that if $a(\lambda) = 1$ and $b(\lambda) = \lambda$, then formula (2.23) reduces to the familiar backward shift realization formula that is valid for rational matrix-valued functions that are analytic at zero; see, e.g., [12] for more information on the latter and additional references.

Formula (2.23) is valid even if the space $\mathcal{R}_c(W)$ is not finite dimensional. If $\mathcal{R}_c(W)$ is finite dimensional, then by Lemma 2.2, there exists a $p \times n$ matrix-valued function

$$F(\lambda) = V\{a(\lambda)A - b(\lambda)B\}^{-1},$$

whose columns form a basis for $\mathcal{R}_c(W)$. Let us assume further that the point α in the definition of the operators $\mathbf{A}, \ldots, \mathbf{D}$ is such that $a(\alpha)A - b(\alpha)B$ is invertible. Then formula (2.8) emerges from (2.23) upon defining U by the recipe

$$\mathbf{U}\eta = F(\lambda)U\eta$$

and carrying out the indicated calculations.

3. MINIMAL REALIZATIONS

Let W be a $p \times q$ matrix-valued function that is meromorphic in Ω and admits a realization of the form (2.8), where α is a point in Ω_W at which $G(\alpha)$ is invertible. Such a realization exists if and only if the spaces $\mathcal{R}_c(W)$ and $\mathcal{R}_r(W)$ are finite dimensional. In this instance, a realization is said to be minimal if the size of $G(\lambda)$ is minimal, i.e., if the common size of the matrices A and B in (2.8) is as small as possible. The number of rows (or columns) in A or B for a minimal realization is termed the ρ-degree of W and is denoted by the symbol $\deg_\rho W$.

LEMMA 3.1. *Let $\rho \in \mathcal{D}_\Omega$ and let W be a $p \times q$ matrix-valued meromorphic function in Ω such that either $\mathcal{R}_c(W)$ or $\mathcal{R}_r(W)$ is finite dimensional. Then the other is finite dimensional and*

$$\deg_\rho W = \dim \mathcal{R}_c(W) = \dim \mathcal{R}_r(W).$$

In particular, the ρ-degree does not depend upon the representation of ρ or on the choice of the point α.

Proof. By Theorems 2.1 and 2.2,

$$\deg_\rho W \leq \dim \mathcal{R}_c(W) = \dim \mathcal{R}_r(W).$$

On the other hand, by (2.13), the inequality cannot be strict. ∎

Let

$$\mu_c = \dim\{VG(\lambda)^{-1}\eta : \eta \in \mathbb{C}^n\} \tag{3.1}$$

and

$$\mu_r = \dim\{\xi^* G(\lambda)^{-1} U : \xi \in \mathbb{C}^n\} \tag{3.2}$$

in the linear space of vector-valued continuous functions on any nonempty open subset Δ of $\{\lambda \in \Omega : \det G(\lambda) \neq 0\}$.

It is clear that the definitions of μ_c and μ_r are independent of the choice of Δ because if say $VG(\lambda)^{-1}\eta = 0$ for every point λ in any such Δ, then the equality propagates to the full domain of analyticity of $G(\lambda)^{-1}$ in Ω: $\{\lambda \in \Omega : \det G(\lambda) \neq 0\}$, since Ω is connected.

We say that the triple $\{V, A, B\}$ is (a,b) observable if $\mu_c = n$ and that the triple $\{A, B, U\}$ is (a,b) controllable if $\mu_r = n$. In other words, $\{V, A, B\}$ is (a,b) observable if and only if the columns of $VG(\lambda)^{-1}$ are linearly independent, whereas the triple $\{A, B, U\}$ is (a,b) controllable if and only if the rows of $G(\lambda)^{-1}U$ are linearly independent.

The following equivalent definition is useful.

LEMMA 3.2. *If $G(\lambda) = a(\lambda)A - b(\lambda)B$ is an $n \times n$ matrix-valued function, and if Δ is any subset of $\{\lambda \in \Omega : \det G(\lambda) \neq 0\}$ that contains infinitely many distinct points including at least one limit point, then:*
The triple $\{V, A, B\}$ is (a, b) observable if and only if

$$\bigcap_{\omega\in\Delta} \operatorname{kernel}\{V(a(\omega)A - b(\omega)B)^{-1}\} = \{0\}. \tag{3.3}$$

The triple $\{A, B, U\}$ is (a, b) controllable if and only if the closed linear span

$$\bigvee_{\omega\in\Delta} \operatorname{range}\{(a(\omega)A - b(\omega)B)^{-1}U\} = \mathbb{C}^n. \tag{3.4}$$

Proof. Clearly

$$\bigcap_{\lambda\in\Delta} \operatorname{kernel}\{VG(\lambda)^{-1}\} = \{\eta \in \mathbb{C}^n : VG(\lambda)^{-1}\eta = 0 \text{ for every point } \lambda \in \Delta\}$$

and

$$\left(\bigvee_{\lambda\in\Delta} \operatorname{range}\{G(\lambda)^{-1}U\}\right)^{\perp} = \{\xi \in \mathbb{C}^n : \xi^* G(\lambda)^{-1}U = 0 \\ \text{for every point } \lambda \in \Delta\}.$$

The descriptions on the right-hand sides of the last two formulas make it evident that the spaces in question are independent of the choice of Δ since $G(\lambda)^{-1}$ is analytic in $\{\lambda \in \Omega : \det G(\lambda) \neq 0\}$. ∎

THEOREM 3.1. *Suppose that $G(\alpha) = a(\alpha)A - b(\alpha)B$ is an invertible $n \times n$ matrix (and hence in particular that $a(\alpha)$ and $b(\alpha)$ are not both simultaneously zero).*

(1) *If $b(\alpha) \neq 0$, then $\{V, A, B\}$ is (a, b) observable if and only if*

$$\bigcap_{j=0}^{n-1} \operatorname{kernel} V\{G(\alpha)^{-1}A\}^j = \{0\}, \tag{3.5}$$

whereas $\{A, B, U\}$ is (a, b) controllable if and only if

$$\operatorname{rank}\left[U \,\vdots\, \{AG(\alpha)^{-1}\}U \,\vdots\, \cdots \,\vdots\, \{AG(\alpha)^{-1}\}^{n-1}U\right] = n. \tag{3.6}$$

(2) *If $a(\alpha) \neq 0$, then $\{V, A, B\}$ is (a, b) observable if and only if*

$$\bigcap_{j=0}^{n-1} \operatorname{kernel} V\{G(\alpha)^{-1}B\}^j = \{0\}, \tag{3.7}$$

whereas $\{A, B, U\}$ is (a, b) controllable if and only if

$$\operatorname{rank}\left[U \vdots \{BG(\alpha)^{-1}\}U \vdots \cdots \vdots \{BG(\alpha)^{-1}\}^{n-1}U\right] = n. \tag{3.8}$$

Proof. Suppose first that $b(\alpha) \neq 0$. Then it is readily checked that

$$B = \{a(\alpha)A - G(\alpha)\}/b(\alpha)$$

and hence that

$$G(\lambda) = \frac{b(\lambda)}{b(\alpha)}G(\alpha) - \frac{\delta_\alpha(\lambda)}{b(\alpha)}A.$$

Now let Δ be a sufficiently small neighborhood of α in Ω such that $b(\lambda) \neq 0$ and $G(\lambda)$ is invertible for $\lambda \in \Delta$. Then for such λ,

$$VG(\lambda)^{-1} = \frac{b(\alpha)}{b(\lambda)}V\left\{I_n - \frac{\delta_\alpha(\lambda)}{b(\lambda)}G(\alpha)^{-1}A\right\}^{-1}G(\alpha)^{-1},$$

from which it follows that the columns of $VG(\lambda)^{-1}$ are linearly independent in the linear space of analytic vector-valued functions in Δ if and only if the same holds true for the columns of

$$V\left\{I_n - \frac{\delta_\alpha(\lambda)}{b(\lambda)}G(\alpha)^{-1}A\right\}^{-1}.$$

But that is equivalent to condition (3.5).

The proof of (3.6) is based on a similar analysis of the linear independence of the rows of $G(\lambda)^{-1}U$ and the identity

$$G(\lambda)^{-1}U = \frac{b(\alpha)}{b(\lambda)}G(\alpha)^{-1}\left\{I_n - \frac{\delta_\alpha(\lambda)}{b(\lambda)}AG(\alpha)^{-1}\right\}^{-1}U.$$

This completes the proof of the assertions for $b(\alpha) \neq 0$. The case $a(\alpha) \neq 0$ is handled in much the same way except that now

$$G(\lambda) = \frac{a(\lambda)}{a(\alpha)}G(\alpha) - \frac{\delta_\alpha(\lambda)}{a(\alpha)}B.$$

The details are left to the reader. ∎

The criteria of Theorem 3.1 reduce to the familiar ones in the classical settings. This is easily seen when $a(\lambda) = 1$ and $b(\lambda) = \lambda$ because then (2.8) yields the realization

$$W(\lambda) = D + (\lambda - \alpha)V(A - \lambda B)^{-1}U.$$

If $G(0) = A$ is invertible, then we may choose $\alpha = 0$ and presume that $A = I_n$. Thus $G(\alpha) = I_n$, and the second part of the theorem (corresponding to $a(\alpha) \neq 0$) supplies the well-known criterion for observability and controllability for realizations of the form

$$W(\lambda) = D + \lambda V(I_n - \lambda B)^{-1}U.$$

On the other hand, if $a(\lambda) = \sqrt{\pi}(\lambda + 1)$, $b(\lambda) = \sqrt{\pi}(\lambda - 1)$, $A = (I_n - T)/2\sqrt{\pi}$, and $B = -(I_n + T)/2\sqrt{\pi}$, then

$$\delta_\alpha(\lambda) = 2\pi(\lambda - \alpha) \qquad \text{and} \qquad G(\lambda) = \lambda I_n - T.$$

Therefore, by (2.8),

$$\begin{aligned} W(\lambda) &= D + 2\pi(\lambda - \alpha)V(\lambda I_n - T)^{-1}U \\ &= D + 2\pi VU - 2\pi V(\lambda I_n - T)^{-1}G(\alpha)U, \end{aligned}$$

which can be reexpressed in the usual form

$$W(\lambda) = D_1 + V(\lambda I_n - T)^{-1}U_1,$$

upon making the self-evident identifications. Suppose further, for the sake of definiteness, that $b(\alpha) \neq 0$. Then clearly:

$$\xi \in \bigcap_{j=0}^{n-1} \ker\{V(G(\alpha)^{-1}A)^j\}$$

$$\iff V\{I - \lambda G(\alpha)^{-1}A\}^{-1}\xi = 0, \qquad \text{for } |\lambda| \leq \varepsilon$$

(for suitably small $\varepsilon > 0$),

$$\iff V\{\alpha I_n - T - \lambda(I_n - T)\}^{-1}G(\alpha)\xi = 0, \qquad \text{for } |\lambda| \leq \frac{\varepsilon}{2\sqrt{\pi}} = \varepsilon_1,$$

$$\iff V\{(\alpha - 1)I_n - (\lambda - 1)(I_n - T)\}^{-1}G(\alpha)\xi = 0, \qquad \text{for } |\lambda| \leq \varepsilon_1,$$

(and hence since $b(\alpha) \neq 0$)

$$\iff V\left\{I_n - \frac{\lambda - 1}{\alpha - 1}(I_n - T)\right\}^{-1} G(\alpha)\xi = 0, \qquad \text{for } |\lambda| \leq \varepsilon_1,$$

$$\iff G(\alpha)\xi \in \bigcap_{j=0}^{n-1} \ker V(I_n - T)^j$$

$$\iff G(\alpha)\xi \in \bigcap_{j=0}^{n-1} \ker VT^j.$$

But this clearly implies that (3.5) holds if and only if

$$\bigcap_{j=0}^{n-1} \ker VT^j = \{0\},$$

i.e., if and only if the realization is observable in the usual sense. The verification that the remaining conclusions of Theorem 3.1 are equivalent to the usual formulations of observability and controllability for the realization under consideration goes through in much the same way.

For additional discussion of assorted definitions of controllability and observability for descriptor systems and related results, see, e.g., [9, 23–26].

THEOREM 3.2. *Let $W(\lambda)$ be a $p\times q$ matrix-valued function that is meromorphic in Ω and admits a realization of the form* (2.8) *where $G(\lambda) \in \mathbb{C}^{n\times n}$ and α is a point in Ω_W at which $G(\alpha)$ is invertible. Then the realization* (2.8) *is minimal if and only if $\mu_r = \mu_c = n$, i.e., if and only if the realization is both (a, b) observable and (a, b) controllable.*

Proof. It follows readily from (2.13) and Lemma 3.2 that

$$\mathcal{R}_c(W) \subseteq \{VG(\lambda)^{-1}\eta : \eta \in \mathbb{C}^n\} \quad \text{and} \quad \mathcal{R}_r(W) \subseteq \{\xi^* G(\lambda)^{-1} U : \xi \in \mathbb{C}^n\},$$

with equality if the realization is both (a, b) observable and (a, b) controllable. Therefore

$$\dim \mathcal{R}_c \leq \mu_c \leq n \qquad \text{and} \qquad \dim \mathcal{R}_r \leq \mu_r \leq n,$$

with equality if the realization is both observable and controllable. The rest is immediate from Lemma 3.1. ∎

THEOREM 3.3. *Let $\rho \in \mathcal{D}_\Omega$ and let W be a $p \times q$ matrix-valued function that admits a pair of minimal realizations of the form* (2.8):

$$W(\lambda) = D_j + \delta_\alpha(\lambda) V_j G_j(\lambda)^{-1} U_j,$$

where

$$G_j(\lambda) = a(\lambda)A_j - b(\lambda)B_j$$

and $G_j(\alpha)$ is an invertible $n \times n$ matrix for $j = 1, 2$. Then

$$D_1 = D_2 = W(\alpha)$$

and there exist a unique pair of $n \times n$ matrices X and Y such that

$$V_2 G_2(\lambda)^{-1} = V_1 G_1(\lambda)^{-1} X \qquad \textit{and} \qquad G_1(\lambda)^{-1} U_1 = Y G_2(\lambda)^{-1} U_2$$

for every $\lambda \in \Omega$ for which both of the indicated inverses exist. For this choice of X and Y

$$V_2 = V_1 Y, \qquad XA_2 = A_1 Y, \qquad XB_2 = B_1 Y, \qquad XU_2 = U_1$$

and so too

$$XG_2(\lambda) = G_1(\lambda) Y,$$

for every point $\lambda \in \Omega$. Moreover, the matrices X and Y are both invertible.

Proof. To begin with, it is useful to recall that

$$\{r(a,b;\alpha)G_j^{-1}\}(\lambda) = G_j(\lambda)^{-1}B_jG_j(\alpha)^{-1} = G_j(\alpha)^{-1}B_jG_j(\lambda)^{-1} \tag{3.9}$$

and

$$\{r(b,a;\alpha)G_j^{-1}\}(\lambda) = -G_j(\lambda)^{-1}A_jG_j(\alpha)^{-1} = -G_j(\alpha)^{-1}A_jG_j(\lambda)^{-1}. \tag{3.10}$$

Under the given assumptions it is clear that

$$W(\alpha) = D_1 = D_2.$$

Therefore

$$V_1G_1(\lambda)^{-1}U_1 = V_2G_2(\lambda)^{-1}U_2 \tag{3.11}$$

for every point $\lambda \in \Omega$ at which the two indicated inverses exist and in particular in a small enough neighborhood Δ of α. Consequently by (3.9),

$$V_1G_1(\lambda)^{-1}\{B_1G_1(\alpha)^{-1}\}^jU_1 = V_2G_2(\lambda)^{-1}\{B_2G_2(\alpha)^{-1}\}^jU_2$$

for $j = 0, 1, \ldots$ and $\lambda \in \Delta$. Thus if $a(\alpha) \neq 0$, it follows from the presumed controllability and (3.8) that

$$\left[U_j \;\vdots\; B_jG_j(\alpha)^{-1}U_j \;\vdots\; \cdots \;\vdots\; \{B_jG_j(\alpha)\}^{n-1}U_j\right]$$

is right invertible for $j = 1, 2$ and hence that there exists an $n \times n$ matrix X such that

$$V_2G_2(\lambda)^{-1} = V_1G_1(\lambda)^{-1}X. \tag{3.12}$$

Moreover, since $V_1G_1(\lambda)^{-1}$ and $V_2G_2(\lambda)^{-1}$ both have n linearly independent columns (in the vector space of continuous vector-valued functions on $\{\lambda \in \Omega : \det G_j(\lambda) \neq 0 \text{ for } j = 1, 2\}$) by the presumed observability, there is only one such matrix X and it is invertible. The same conclusion holds even if $a(\alpha) = 0$, because then $b(\alpha) \neq 0$ and a similar set of formulas is obtained but with A_j in place of B_j upon applying (3.10) to (3.11).

In much the same way it follows from (3.9)–(3.11) that

$$V_1\{G_1(\alpha)^{-1}B_1\}^j G_1(\lambda)^{-1}U_1 = V_2\{G_2(\alpha)^{-1}B_2\}^j G_2(\lambda)^{-1}U_2$$

and

$$V_1\{G_1(\alpha)^{-1}A_1\}^j G_1(\lambda)^{-1}U_1 = V_2\{G_2(\alpha)^{-1}A_2\}^j G_2(\lambda)^{-1}U_2$$

for $j = 0, \dots, n-1$ and hence, by the presumed observability and Theorem 3.1, that there exists an $n \times n$ matrix Y such that

$$G_1(\lambda)^{-1}U_1 = YG_2(\lambda)^{-1}U_2. \tag{3.13}$$

Since the rows in each of the matrix-valued functions $G_1(\lambda)^{-1}U_1$ and $G_2(\lambda)^{-1}U_2$ are linearly independent by the presumed controllability, there is only one such Y and it is invertible.

Now, upon substituting (3.12) and (3.13) into the formula

$$V_1G_1(\lambda)^{-1}G_1(\alpha)G_1(\omega)^{-1}U_1 = V_2G_2(\lambda)^{-1}G_2(\alpha)G_2(\omega)^{-1}U_2$$

(which may be obtained from (2.13)), and exploiting the linear independence of the columns of $V_1G_1(\lambda)^{-1}$ and the linear independence of the rows of $G_2(\lambda)^{-1}U_2$, it is readily seen that

$$Y = G_1(\alpha)^{-1}XG_2(\alpha).$$

Therefore $V_2 = V_1Y$ by (3.12) and $U_2 = X^{-1}U_1$ by (3.13). Finally, successive applications of (3.9) and (3.10) to (3.12) lead easily to the conclusion that $A_2 = X^{-1}A_1Y$ and $B_2 = X^{-1}B_1Y$.

Finally the asserted uniqueness follows from the presumed minimality and the fact that a pair of $n \times n$ invertible matrices X and Y meet the four stated conditions $V_2 = V_1Y, \dots, XU_2 = U_1$, if and only if (3.12) and (3.13) hold. ∎

We remark that such connections between minimal realizations have been observed earlier for assorted subclasses of the realizations considered in Theorem 3.3; see, e.g., [9, 25, 26], and formula (2.3) of [13].

THEOREM 3.4. *Let W be a $p \times q$ matrix-valued function that admits a minimal realization*

$$W(\lambda) = D + \delta_\alpha(\lambda)V\{a(\lambda)A - b(\lambda)B\}^{-1}U$$

with $\deg_\rho W = n$ *in which* $a(\alpha)A - b(\alpha)B$ *is invertible. If* $W(\lambda)$ *admits a second realization*

$$W(\lambda) = D_1 + \{c(\alpha)d(\lambda) - d(\alpha)c(\lambda)\}V_1\{c(\lambda)A_1 - d(\lambda)B_1\}^{-1}U_1$$

in terms of a pair of functions $c(\lambda)$ *and* $d(\lambda)$ *as in* (1.3) *and if* $c(\alpha)A_1 - d(\alpha)B_1$ *is invertible, then* $D_1 = D$ *and this second realization is minimal if and only if there exists a pair of* $n \times n$ *invertible matrices* X *and* Y *such that*

$$\begin{gathered}(\det M)V_1 = VY, \qquad m_{11}A_1 - m_{21}B_1 = X^{-1}AY, \\ m_{22}B_1 - m_{21}A_1 = X^{-1}BY \qquad \text{and} \qquad U_1 = X^{-1}U,\end{gathered}$$

where the m_{ij} *are the entries in the matrix* M *which intervenes in* (1.4).

Proof. By (1.4),

$$\begin{aligned} c(\lambda)A_1 - d(\lambda)B_1 &= \begin{bmatrix} c(\lambda)I_n & d(\lambda)I_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \\ &= \begin{bmatrix} a(\lambda)I_n & b(\lambda)I_n \end{bmatrix} \begin{bmatrix} m_{11}A_1 - m_{12}B_1 \\ m_{21}A_1 - m_{22}B_1 \end{bmatrix}. \end{aligned}$$

Therefore by (1.6),

$$\begin{aligned} &V\{a(\lambda)A - b(\lambda)B\}^{-1}U \\ &\quad = (\det M)V_1\{a(\lambda)[m_{11}A_1 - m_{12}B_1] - b(\lambda)[m_{22}B_1 - m_{21}A_1]\}^{-1}U_1. \end{aligned}$$

The rest is immediate from Theorem 3.3. ■

We turn now to an analogue in the present framework of another characterization of observability and controllability that is useful in the classical setting; for the latter see [14].

THEOREM 3.5. *Assume that* $\rho_\omega(\lambda) = a(\lambda)a(\omega)^* - b(\lambda)b(\omega)^*$ *belongs to* $\mathcal{D}_\Omega$ *and that* $G(\lambda) = a(\lambda)A - b(\lambda)B$ *is an* $n \times n$ *matrix-valued function.*

I. *If*

$$\{\lambda \in \Omega : a(\lambda) = 0 \quad \text{and} \quad b(\lambda) \neq 0\} \neq \emptyset, \tag{3.14}$$

then:

I.1. *The triple* $\{A, B, U\}$ *is* (a, b) *controllable if and only if*

$$\operatorname{rank}[A - \gamma B \quad U] = n \tag{3.15}$$

for every point γ *in the set*

$$\Gamma_1 = \mathbb{C} \setminus \left\{ \frac{b(\lambda)}{a(\lambda)} : \lambda \in \Omega \quad \textit{and} \quad a(\lambda) \neq 0 \right\}$$

and

$$\operatorname{rank}[G(\lambda) \quad U] = n \tag{3.16}$$

for every point $\lambda \in \Omega$ *for which*

$$|a(\lambda)| + |b(\lambda)| > 0. \tag{3.17}$$

I.2. *The triple* $\{V, A, B\}$ *is* (a, b) *observable if and only if*

$$\operatorname{kernel} \begin{bmatrix} A - \gamma B \\ V \end{bmatrix} = \{0\} \tag{3.18}$$

for every point $\gamma \in \Gamma_1$ *and*

$$\operatorname{kernel} \begin{bmatrix} G(\lambda) \\ V \end{bmatrix} = \{0\} \tag{3.19}$$

for every point $\lambda \in \Omega$ *for which* (3.17) *holds.*

II. *If*

$$\{\lambda \in \Omega : b(\lambda) = 0 \quad \textit{and} \quad a(\lambda) \neq 0\} \neq \emptyset, \tag{3.20}$$

then:

II.1. *The triple* $\{A, B, U\}$ *is* (a, b) *controllable if and only if*

$$\operatorname{rank}[\gamma A - B \quad U] = n \qquad \textit{and} \qquad \operatorname{rank}[G(\lambda) \quad U] = n \tag{3.21}$$

for every point γ *in the set*

$$\Gamma_2 = \mathbb{C} \setminus \left\{ \frac{a(\lambda)}{b(\lambda)} : \lambda \in \Omega \quad \textit{and} \quad b(\lambda) \neq 0 \right\}$$

and every point $\lambda \in \Omega$ *at which* (3.17) *holds, respectively.*

II.2. *The triple $\{V, A, B\}$ is (a,b) observable if and only if*

$$\text{kernel}\begin{bmatrix} \gamma A - B \\ V \end{bmatrix} = \{0\} \qquad \textit{and} \qquad \text{kernel}\begin{bmatrix} G(\lambda) \\ V \end{bmatrix} = \{0\} \quad (3.22)$$

for every point $\gamma \in \Gamma_2$ and every point $\lambda \in \Omega$ at which (3.17) *holds, respectively.*

Proof. We shall prove only I.1 since the remaining assertions may be established in much the same way. The proof is divided into steps.

Step 1. If the triple $\{A, B, U\}$ is (a,b) controllable, then (3.16) holds for every point $\lambda \in \Omega$ that meets (3.17).

Proof of Step 1. Suppose that $\{A, B, U\}$ is (a,b) controllable, and let u be any vector in $\mathbb{C}^n$ such that

$$u^*G(\beta) = 0 \qquad \text{and} \qquad u^*U = 0$$

for some point $\beta \in \Omega$ that meets condition (3.17). Then

$$a(\beta)u^*G(\alpha) = \delta_\alpha(\beta)u^*B \qquad (\text{and}\, b(\beta)u^*G(\alpha) = \delta_\alpha(\beta)u^*A).$$

Now choose a point $\alpha \in \Omega$ such that $G(\alpha)$ is invertible and $a(\alpha) \neq 0$ and observe that if $\delta_\alpha(\beta) = 0$, then the conditions $|a(\beta)| + |b(\beta)| > 0$ and $a(\alpha) \neq 0$ guarantee that $a(\beta) \neq 0$ and hence that $u = 0$. On the other hand, if $\delta_\alpha(\beta) \neq 0$, then

$$u^*\{BG(\alpha)^{-1}\}^j U = \left\{\frac{a(\beta)}{\delta_\alpha(\beta)}\right\}^j u^*U = 0$$

for $j = 0, 1, \ldots$. Therefore the presumed controllability, as expressed in (3.8), forces $u = 0$ in this case also. This completes the proof of Step 1.

Step 2. If the triple $\{A, B, U\}$ is (a,b) controllable, then (3.15) holds for every point $\gamma \in \Gamma_1$.

Proof of Step 2. Let $u \in \mathbb{C}^n$ be such that

$$u^*[A - \gamma B \quad U] = 0.$$

Then

$$u^*G(\alpha) = \{a(\alpha)\gamma - b(\alpha)\}u^*B,$$

for every point $\alpha \in \Omega$, and hence, upon choosing α so that $a(\alpha) \neq 0$ and $G(\alpha)$ is invertible, it follows readily that $a(\alpha)\gamma - b(\alpha) \neq 0$ by the choice of γ and that

$$u^*\{BG(\alpha)^{-1}\}^j U = \{a(\alpha)\gamma - b(\alpha)\}^{-j} u^* U = 0$$

for $j = 0, \ldots, n-1$. Therefore $u = 0$ by the presumed controllability.

Step 3. If $\{A, B, U\}$ is not (a, b) controllable and if α is a point in Ω at which $a(\alpha) \neq 0$ and $G(\alpha)$ is invertible, then there exists a nonzero vector $v \in \mathbb{C}^n$ such that

$$v^*U = 0 \quad \text{and} \quad \{1 + \beta b(\alpha)\}v^*B = \beta a(\alpha)v^*A, \tag{3.23}$$

for some point $\beta \in \mathbb{C}$.

Proof of Step 3. Fix a point $\alpha \in \Omega$ such that $G(\alpha)$ is invertible and $a(\alpha) \neq 0$. Then, by the negation of (3.8), there exists a nonzero vector $u \in \mathbb{C}^n$ such that

$$u^*\{BG(\alpha)^{-1}\}^j U = 0$$

for $j = 0, \ldots, n-1$. Let

$$\mathcal{N}_u = \operatorname{span}\{u^*, u^*BG(\alpha)^{-1}, \ldots, u^*\{BG(\alpha)^{-1}\}^{n-1}\}.$$

Then, since $\mathcal{N}_u BG(\alpha)^{-1} \subseteq \mathcal{N}_u$ and $\mathcal{N}_u \neq \{0\}$, there exists a nonzero vector $v^* \in \mathcal{N}_u$ such that

$$v^*BG(\alpha)^{-1} = \beta v^*$$

(and of course $v^*U = 0$ by the choice of u). But this is clearly equivalent to (3.23).

Step 4. If (3.14) and the two rank conditions (3.15) and (3.16) hold, then the triple $\{A, B, U\}$ is controllable.

Proof of Step 4. If $\{A, B, U\}$ is not controllable, then by Step 3 there exists a nonzero vector $v \in \mathbb{C}^n$ and a point $\alpha \in \Omega$ with $a(\alpha) \neq 0$ such that (3.23) holds for some point $\beta \in \mathbb{C}$. Now if $\beta = 0$, then $v^*B = 0$. But by (3.14), there exists a point $\omega \in \Omega$ such that $a(\omega) = 0$ and $b(\omega) \neq 0$. Therefore

$$v^*[G(\omega) \quad U] = 0,$$

which, by (3.16), is only viable if $v = 0$. Next, if $\beta \neq 0$, then

$$v^*A = \frac{1+\beta b(\alpha)}{\beta a(\alpha)} v^*B,$$

which again is only viable if $v = 0$ by either (3.16) or (3.15), since either

$$\frac{1+\beta b(\alpha)}{\beta a(\alpha)} = \frac{b(\omega)}{a(\omega)}$$

for some point $\omega \in \Omega$ with $a(\omega) \neq 0$ or

$$\frac{1+\beta b(\alpha)}{\beta a(\alpha)} = \gamma$$

for some point $\gamma \in \Gamma_1$. This completes the proof of Step 4, and so too the proof of I.1. ∎

COROLLARY. *If either* (3.14) *or* (3.20) *holds, then:*

(1) *The triple* $\{A, B, U\}$ *is* (a, b) *controllable if and only if*

$$\operatorname{rank}[B \quad U] = n \qquad \text{and} \qquad \operatorname{rank}[A - \lambda B \quad U] = n$$

for every point $\lambda \in \mathbb{C}$.

(2) *The triple* $\{V, A, B\}$ *is* (a, b) *observable if and only if*

$$\operatorname{kernel}\begin{bmatrix} B \\ V \end{bmatrix} = \{0\} \qquad \text{and} \qquad \operatorname{kernel}\begin{bmatrix} A - \lambda B \\ V \end{bmatrix} = 0$$

for every point $\lambda \in \mathbb{C}$.

Our next objective is to show that for minimal realizations of W, the domain of analyticity Ω_W of W in Ω

$$\Omega_W = \{\lambda \in \Omega : \det G(\lambda) \neq 0\}. \tag{3.24}$$

This is the natural generalization to the present setting of the well-known fact that for a rational matrix-valued function $F_1(\lambda)$ with minimal realization (in traditional notation)

$$F_1(\lambda) = D_1 + C_1(\lambda I_n - A_1)^{-1}B_1,$$

the poles of $F_1(\lambda)$ coincide with the spectrum of A_1. It is convenient to first establish a preliminary lemma. The proof is adapted from the proof of Theorem 3.1 in [10].

LEMMA 3.3. *Let $V \in \mathbb{C}^{p\times n}$, let $G(\lambda) = a(\lambda)A - b(\lambda)B$ be an $n \times n$ matrix-valued function with* $\det G(\lambda) \not\equiv 0$, *and assume that the $p\times n$ matrix-valued function $F(\lambda) = VG(\lambda)^{-1}$ has n linearly independent columns and is analytic at a point $\omega \in \Omega$ where $a(\omega)$ and $b(\omega)$ are not both zero. Then $G(\omega)$ is invertible.*

Proof. Let $\xi \in \ker G(\omega)$. Then

$$F(\lambda)G(\lambda)\xi = V\xi = 0,$$

as follows by first choosing $\lambda = \omega$.

Suppose now that $a(\omega) \neq 0$. Then

$$a(\lambda)F(\lambda)G(\omega)\xi = 0$$

and

$$a(\omega)F(\lambda)G(\lambda)\xi = 0,$$

and hence, as follows readily upon subtracting the second equation from the first,

$$\delta_\omega(\lambda)F(\lambda)B\xi = 0.$$

Since the zeros of the analytic function $\delta_\omega(\lambda)$ are isolated, it follows further that

$$F(\lambda)B\xi = 0.$$

Therefore, by the presumed linear independence of the columns of F, $B\xi = 0$, whereas $G(\omega)\xi = 0$ by assumption. This in turn implies that $A\xi = 0$ (this is where the assumption $a(\omega) \neq 0$ comes into play) and hence that $\xi \in \ker A \cap \ker B = \{0\}$. This proves that $\ker G(\omega) = \{0\}$ if $a(\omega) \neq 0$. A similar argument leads to the same conclusion if $b(\omega) \neq 0$. Therefore $G(\omega)$ is invertible, as claimed. ■

For the sake of completeness, we add the companion "row result", which is proved in much the same way.

LEMMA 3.4. *If* $\det G(\lambda) \not\equiv 0$ *and if the $n \times q$ matrix-valued function $G(\lambda)^{-1}U$ has n linearly independent rows and is analytic at a point $\omega \in \Omega$ where $a(\omega)$ and $b(\omega)$ are not both zero, then $G(\omega)$ is invertible.*

THEOREM 3.6. *Let W be a $p \times q$ matrix-valued function that admits a realization of the form (2.8) in which $G(\lambda) = a(\lambda)A - b(\lambda)B$ is of minimal size (i.e., $n \times n$ if $\deg_\rho W = n$) and is invertible at α. Then $W(\lambda)$ is analytic at a point $\omega \in \Omega$ such that $|a(\omega)| + |b(\omega)| > 0$ if and only if $\det G(\omega) \neq 0$.*

Proof. In view of Lemma 3.3 (which is applicable because of the presumed (a, b) observability of the realization), it suffices to show that the domains of analyticity Ω_F of F in Ω and Ω_W of W in Ω coincide: $\Omega_F = \Omega_W$. Since the inclusion $\Omega_F \subseteq \Omega_W$ is self-evident, it remains only to establish the reverse inclusion. The proof rests on the presumed (a, b) controllability of the realization. By (2.13),

$$W(\lambda) = W(\beta) + \delta_\beta(\lambda)F(\lambda)G(\alpha)G(\beta)^{-1}U$$

for every pair of points λ and β in Ω_F that meet condition (3.17). Let $\omega \in \Omega_W$. Then there exists an open deleted neighborhood Δ'_ω of ω inside Ω such that $\delta_\omega(\lambda) \neq 0$ and $G(\lambda)$ is invertible for every point $\lambda \in \Delta'_\omega$. Therefore, since $\delta_\beta(\omega) \neq 0$ for $\beta \in \Delta'_\omega$, there exists an $\varepsilon_\beta > 0$ such that $\delta_\beta(\lambda) \neq 0$ for $|\lambda - \omega| \leq \varepsilon_\beta$. Thus

$$\frac{W(\lambda) - W(\beta)}{\delta_\beta(\lambda)} = F(\lambda)G(\alpha)G(\beta)^{-1}U$$

is an analytic function of λ for $|\lambda - \omega| < \varepsilon_\beta$. Fix such a point β_0 and choose a sufficiently small open neighborhood Γ of β_0 such that $G(\beta)$ is invertible for $\beta \in \Gamma$. Then by Lemma 3.2 and the presumed controllability, there exists a set of points $\beta_1, \ldots, \beta_k$ and vectors $\eta_1, \ldots, \eta_k$ in $\mathbb{C}^q$ with $k \leq n$ such that

$$\sum_{j=1}^{k} G(\beta_j)^{-1}U\eta_j = G(\alpha)x$$

for any preassigned choice of $x \in \mathbb{C}^n$. But this implies that

$$F(\lambda)x = \sum_{j=1}^{k} F(\lambda)G(\alpha)G(\beta_j)^{-1}U\eta_j$$

is an analytic function of λ for

$$|\lambda - \omega| < \min\{\varepsilon_{\beta_j}, j = 1, \ldots, k\}.$$

Since x is arbitrary it follows that $\omega \in \Omega_F$, as claimed. This completes the proof. ∎

4. REPRODUCING KERNEL SPACES OF PAIRS

In the sequel we are particularly interested in matrix-valued functions $W(\lambda) = D + \delta_\alpha(\lambda)VG(\lambda)^{-1}U$ that are J unitary with respect to a given signature matrix J on Ω_0:

$$W(\lambda)^*JW(\lambda) = J$$

for $\lambda \in \Omega_0$. In general there is no a priori reason why a given $W(\lambda)$ should meet this condition unless additional constraints are imposed on the matrices D, V, A, B, and U, which figure in the realization. This theme is developed and expanded upon (to include $W(\lambda)$ which are isometric, $W^*\Sigma_p W = \Sigma_q$, or coisometric, $W\Sigma_q W^* = \Sigma_p$, on Ω_0 with respect to a given pair of signature matrices Σ_p and Σ_q) in the next section. Nevertheless, it is always possible to "pair" any such square matrix-valued function $W(\lambda)$ with nonidentically vanishing determinant, with an associated partner $W_\pi(\lambda)$ such that

$$W(\lambda)JW_\pi(\lambda)^* = J$$

for $\lambda \in \Omega_0$. The theory that underlies such pairings rests on the theory of finite-dimensional pair spaces. This is reviewed in Section 4.1.

Sections 4.2 and 4.3 deal with the construction of an associated partner $W_\pi(\lambda)$ for $W(\lambda)$ and some operator theoretical interpretations, respectively.

4.1. Reproducing Kernel Spaces of Pairs

Let $\mathcal{M}_{\mathrm{L}}$ and $\mathcal{M}_{\mathrm{R}}$ be finite-dimensional vector spaces of $m \times 1$ vector-valued functions that are analytic in an open nonempty subset Δ of $\mathbb{C}$. The space $\mathcal{M} = \mathcal{M}_{\mathrm{L}} \times \mathcal{M}_{\mathrm{R}}$ is said to be a reproducing kernel pair space (or space of pairs) with respect to a sesquilinear form $[\ ,\ ]_{\mathcal{M}}$ if there exist a pair $(K_\omega^{\mathrm{L}}(\lambda), K_\omega^{\mathrm{R}}(\lambda))$ of $m \times m$ matrix-valued functions such that for every choice of $\omega \in \Delta$ and $u \in \mathbb{C}^m$ the following two conditions hold:

(1) $K_\omega^{\mathrm{R}}u \in \mathcal{M}_{\mathrm{R}}$ and

$$\left[g, K_\omega^{\mathrm{R}}u\right]_{\mathcal{M}} = u^*g(\omega) \tag{4.1}$$

for every $g \in \mathcal{M}_{\mathrm{L}}$.

(2) $K_\omega^{\mathrm{L}}u \in \mathcal{M}_{\mathrm{L}}$ and

$$\left[K_\omega^{\mathrm{L}}u, \quad h\right]_{\mathcal{M}} = h(\omega)^*u \tag{4.2}$$

for every $h \in \mathcal{M}_{\mathrm{R}}$.

It follows readily from (4.1) and (4.2) that:

(1) The reproducing kernel pair is unique; i.e., if $(M_\omega^{\mathrm{L}}, M_\omega^{\mathrm{R}})$ is a second reproducing kernel pair for $\mathcal{M}$, then

$$M_\omega^{\mathrm{L}} = K_\omega^{\mathrm{L}} \qquad \text{and} \qquad M_\omega^{\mathrm{R}} = K_\omega^{\mathrm{R}}.$$

(2) $K_\alpha^{\mathrm{L}}(\beta) = K_\beta^{\mathrm{R}}(\alpha)^*$.
(3) $\dim \mathcal{M}_{\mathrm{L}} = \dim \mathcal{M}_{\mathrm{R}}$.

For ease of exposition we have formulated the definitions for vector-valued functions that are analytic in Δ. The same definitions work for vector-valued functions that are meromorphic in Δ except that then equality is only required at points of analyticity.

In the present finite-dimensional setting $\mathcal{M}_{\mathrm{L}}$ [resp. $\mathcal{M}_{\mathrm{R}}$] is generated by the span of the columns of an $m \times n$ matrix-valued function F_{L} [resp. F_{R}] with linearly independent columns. Then for any invertible $n \times n$ matrix P, Hermitian or not, the space $\mathcal{M} = \mathcal{M}_{\mathrm{L}} \times \mathcal{M}_{\mathrm{R}}$ endowed with the sesquilinear product

$$[F_{\mathrm{L}}\xi, \quad F_{\mathrm{R}}\eta]_{\mathcal{M}} = \eta^* P \xi \tag{4.3}$$

is readily seen to be a reproducing kernel pair space with

$$K_\omega^{\mathrm{L}}(\lambda) = F_{\mathrm{L}}(\lambda) P^{-1} F_{\mathrm{R}}(\omega)^* \tag{4.4}$$

and

$$K_\omega^{\mathrm{R}}(\lambda) = F_{\mathrm{R}}(\lambda) P^{-*} F_{\mathrm{L}}(\omega)^*. \tag{4.5}$$

From these formulas it also follows that

$$\mathcal{M}_{\mathrm{L}} = \operatorname{span}\{K_\omega^{\mathrm{L}}\xi\} \qquad \text{and} \qquad \mathcal{M}_{\mathrm{R}} = \operatorname{span}\{K_\omega^{\mathrm{R}}\xi\},$$

where ξ runs over $\mathbb{C}^p$ and ω varies over a nonempty open subset of the domain of analyticity of F_{L} and F_{R} in Δ.

It is important to bear in mind that even though $K_\omega^{\mathrm{L}}\xi \in \mathcal{M}_{\mathrm{L}}$ [resp. $K_\omega^{\mathrm{R}}\xi \in \mathcal{M}_{\mathrm{R}}$], it is not the reproducing kernel for $\mathcal{M}_{\mathrm{L}}$ [resp. $\mathcal{M}_{\mathrm{R}}$]. In fact, if

$$\langle F_{\mathrm{L}}\xi, F_{\mathrm{L}}\eta\rangle_{\mathcal{M}_{\mathrm{L}}} = \eta^* P_{\mathrm{L}} \xi$$

and

$$\langle F_{\mathrm{R}}\xi, F_{\mathrm{R}}\eta\rangle_{\mathcal{M}_{\mathrm{R}}} = \eta^* P_{\mathrm{R}} \xi$$

for some pair of positive definite matrices $P_{\rm L}$ and $P_{\rm R}$, then the reproducing kernels for $\mathcal{M}_{\rm L}$ and $\mathcal{M}_{\rm R}$ are equal to

$$F_{\rm L}(\lambda)P_{\rm L}^{-1}F_{\rm L}(\omega)^* \qquad \text{and} \qquad F_{\rm R}(\lambda)P_{\rm R}^{-1}F_{\rm R}(\omega)^*,$$

respectively. These expressions will not agree with $K_\omega^{\rm L}(\lambda)$ and $K_\omega^{\rm R}(\lambda)$, respectively, unless $P^{-*}P_{\rm L} = P_{\rm R}^{-1}P$ and $F_{\rm L} = F_{\rm R}P_{\rm R}^{-1}P$.

THEOREM 4.1. *Let $\rho_\omega(\lambda) = a(\lambda)a(\omega)^* - b(\lambda)b(\omega)^*$ belong to $\mathcal{D}_\Omega$ and let $\mathcal{M}_{\rm L}$ and $\mathcal{M}_{\rm R}$ be n-dimensional vector spaces of $m \times 1$ vector-valued meromorphic functions in Ω that are spanned by the columns of the $m \times n$ matrix-valued functions*

$$F_{\rm L}(\lambda) = V_{\rm L}\{a(\lambda)A_{\rm L} - b(\lambda)B_{\rm L}\}^{-1} \quad \textit{and} \quad F_{\rm R}(\lambda) = V_{\rm R}\{a(\lambda)A_{\rm R} - b(\lambda)B_{\rm R}\}^{-1},$$

respectively. Let P be an invertible $n \times n$ matrix (Hermitian or not), and let J be any $m \times m$ signature matrix (i.e., $J = J^*$ and $JJ^* = I_m$). Then the space $\mathcal{M} = \mathcal{M}_{\rm L} \times \mathcal{M}_{\rm R}$ endowed with the sesquilinear form (4.3) is a reproducing kernel pair space with reproducing kernel pair $(K_\omega^{\rm L}(\lambda), K_\omega^{\rm R}(\lambda))$ specified by (4.4) and (4.5). The kernels $K_\omega^{\rm L}(\lambda)$ and $K_\omega^{\rm R}(\lambda)$ can be expressed in the form

$$K_\omega^{\rm L}(\lambda) = \frac{J - W_{\rm L}(\lambda)JW_{\rm R}(\omega)^*}{\rho_\omega(\lambda)} \tag{4.6}$$

and

$$K_\omega^{\rm R}(\lambda) = \frac{J - W_{\rm R}(\lambda)JW_{\rm L}(\omega)^*}{\rho_\omega(\lambda)}, \tag{4.7}$$

respectively, if and only if P is a solution of the matrix equation

$$A_{\rm R}^*PA_{\rm L} - B_{\rm R}^*PB_{\rm L} = V_{\rm R}^*JV_{\rm L}. \tag{4.8}$$

The $m \times m$ matrix-valued functions $W_{\rm L}(\lambda)$ and $W_{\rm R}(\lambda)$, which intervene in (4.6) and (4.7) are meromorphic in Ω. If μ and μ' are reflections (i.e., $\rho_\mu(\mu') = 0$) such that $F_{\rm L}$ is analytic at μ and $F_{\rm R}$ is analytic at μ', then $W_{\rm L}$ and $W_{\rm R}$ are uniquely specified by the formulas

$$W_{\rm L}(\lambda) = \{I_m - \rho_{\mu'}(\lambda)F_{\rm L}(\lambda)P^{-1}F_{\rm R}(\mu')^*J\}C_{\rm L} \tag{4.9}$$

and

$$W_{\rm R}(\lambda) = \{I_m - \rho_\mu(\lambda)F_{\rm R}(\lambda)P^{-*}F_{\rm L}(\mu)^*J\}C_{\rm R} \tag{4.10}$$

up to a pair of constant $m \times m$ matrix factors $C_{\rm L} = W_{\rm L}(\mu)$ and $C_{\rm R} = W_{\rm R}(\mu')$ on the right, which are subject to the constraint

$$C_{\rm L} J C_{\rm R}^* = J. \tag{4.11}$$

Theorem 4.1 and more general versions thereof are established in [7].

4.2. Pairing

In this subsection we show how to associate a reproducing kernel space of pairs with every $p \times p$ matrix-valued meromorphic function W in Ω with $\det W(\lambda) \not\equiv 0$ that admits a realization of the form (2.8). In particular, we deduce a formula for W_π in terms of the components in the realization (2.8) for W; W and W_π may then be identified with the pair $W_{\rm L}$ and $W_{\rm R}$, in either order.

LEMMA 4.1. *Let W be a $p \times p$ matrix-valued meromorphic function in Ω that admits a realization of the form* (2.8) *for some point $\alpha \in \Omega_W$ at which both $D = W(\alpha)$ and $G(\alpha)$ are invertible. Let*

$$\widehat{A} = A - b(\alpha) U D^{-1} V, \tag{4.12}$$

$$\widehat{B} = B - a(\alpha) U D^{-1} V, \tag{4.13}$$

and

$$\widehat{G}(\lambda) := a(\lambda)\widehat{A} - b(\lambda)\widehat{B} = G(\lambda) + \delta_\alpha(\lambda) U D^{-1} V. \tag{4.14}$$

Then

$$\widehat{G}(\alpha) = G(\alpha) \tag{4.15}$$

is invertible. Moreover:

(1) *$\widehat{G}(\lambda)$ is invertible and*

$$W(\lambda)^{-1} = D^{-1} - \delta_\alpha(\lambda) D^{-1} V \widehat{G}(\lambda)^{-1} U D^{-1} \tag{4.16}$$

at every point $\lambda \in \Omega_W$ at which both $W(\lambda)$ and $G(\lambda)$ are invertible.

(2) *The realization* (4.16) *for $W(\lambda)^{-1}$ is minimal if and only if the realization* (2.8) *for $W(\lambda)$ is minimal.*

Proof. The first main assertion is immediate from Lemma 2.6, upon making the requisite identifications. The second assertion is a standard argument that rests on the fact that $\widehat{G}(\lambda)$ and $G(\lambda)$ are matrix-valued functions of the same size and $\{W(\lambda)^{-1}\}^{-1} = W(\lambda)$. ■

If $a(\lambda) = 1$ and $b(\lambda) = \lambda$, then formula (4.16) agrees with formula (2.2) of [13], as it should. If also $\alpha = 0$, then $\widehat{G}(0) = \widehat{A} = A$ is invertible and hence the formulas for W and W^{-1} can be reexpressed in the more familiar forms

$$W(\lambda) = D + \lambda K(I - \lambda M)^{-1}U,$$

and

$$W(\lambda)^{-1} = D^{-1} - \lambda D^{-1}K(I - \lambda M^{\times})^{-1}UD^{-1},$$

where

$$K = VA^{-1}, \qquad M = BA^{-1}, \qquad M^{\times} = \widehat{B}A^{-1} = M - UD^{-1}K.$$

A point $\lambda' \in \Omega$ is said to be a reflection of the point $\lambda \in \Omega$ with respect to ρ if $\rho_{\lambda'}(\lambda) = 0$. In this general setting a point may have many reflections. A subset Δ of Ω is said to be symmetric with respect to ρ if every point $\lambda \in \Delta$ has at least one reflection in Δ. For the classical choices $\rho_\omega(\lambda) = 1 - \lambda\omega^*$ and $\rho_\omega(\lambda) = -2\pi i(\lambda - \omega^*)$, $\lambda' = 1/\lambda^*$ (if $\lambda \neq 0$) and $\lambda' = \lambda^*$, respectively. For these choices of $\rho_\omega(\lambda)$, every point $\lambda \in \mathbb{C}$ (except $\lambda = 0$ in the first case) has exactly one reflection. On the other hand, if $\rho_\omega(\lambda) = -2\pi i(\lambda - \omega^*)(1 - \lambda\omega^*)$, then every point $\lambda \in \mathbb{C}$ except $\lambda = 0$ and $\lambda = \pm 1$ has exactly two reflections: $\lambda = 1/\lambda^*$ and $\lambda = \lambda^*$. If $\lambda = \pm 1$ then these two reflections coincide. For more information on this example see [6]. It is perhaps worth emphasizing that even though a point $\lambda \in \Omega$ may have several reflections with respect to $\rho_\omega(\lambda)$, the ratio $a(\lambda)/b(\lambda) = a(\lambda')/b(\lambda')$ (when meaningful) is independent of the choice of λ'.

THEOREM 4.2. *Let W be a $p \times p$ matrix-valued meromorphic function in Ω that admits a realization of the form (2.8) for some point $\alpha \in \Omega_W$ at which both $D = W(\alpha)$ and $G(\alpha)$ are invertible. Suppose further that α admits a reflection $\alpha' \in \Omega$ such that*

$$a(\alpha)a(\alpha')^* = b(\alpha)b(\alpha')^* \neq 0,$$

let J be a signature matrix, and let

$$D_\pi = JD^{-*}J, \qquad V_\pi = JD^{-*}U^*, \qquad U_\pi = V^*D^{-*}J,$$
$$A_\pi = \frac{b(\alpha')}{a(\alpha)^*}\widehat{B}^*, \qquad B_\pi = \frac{b(\alpha')}{a(\alpha)^*}\widehat{A}^*, \qquad G_\pi(\lambda) = a(\lambda)A_\pi - b(\lambda)B_\pi.$$

Then the matrix-valued function

$$W_\pi(\lambda) = D_\pi - \delta_{\alpha'}(\lambda)V_\pi G_\pi(\lambda)^{-1}U_\pi \tag{4.17}$$

is a partner for $W(\lambda)$:

$$W(\lambda)JW_\pi(\lambda')^* = J$$

for every point $\lambda \in \Omega$ *that admits a reflection* $\lambda' \in \Omega$ *such that* $W(\lambda)$ *and* $G(\lambda)$ *are both invertible, and* $\delta_\alpha(\lambda)$ *and* $\delta_{\alpha'}(\lambda')$ *are both nonzero. (In particular, we may take* $\lambda' = \lambda$ *for* $\lambda \in \Omega_0$.)

Proof. The definition of partner implies that

$$W_\pi(\lambda) = JW(\lambda')^{-*}J$$

providing that the indicated inverse exists. Granting this for the moment, it is then clear from the last lemma that

$$W_\pi(\lambda) = D_\pi - \delta_\alpha(\lambda')^* V_\pi \widehat{G}(\lambda')^{-*} U_\pi. \tag{4.18}$$

Next, by repeated use of the identities

$$a(\alpha)a(\alpha')^* = b(\alpha)b(\alpha')^* \qquad \text{and} \qquad a(\lambda)a(\lambda')^* = b(\lambda)b(\lambda')^*,$$

it is readily checked that

$$\begin{aligned} \delta_\alpha(\lambda')^* a(\lambda) A_\pi &= -\delta_{\alpha'}(\lambda) b(\lambda')^* \widehat{B}^*, \\ \delta_\alpha(\lambda')^* b(\lambda) B_\pi &= -\delta_{\alpha'}(\lambda) a(\lambda')^* \widehat{A}^*, \end{aligned}$$

and hence that

$$\delta_\alpha(\lambda')^* G_\pi(\lambda) = \delta_{\alpha'}(\lambda)\widehat{G}(\lambda')^*. \tag{4.19}$$

The desired formula (4.17) now follows easily from (4.18) and (4.19).

Now, having obtained a formula (or more precisely a candidate for a formula) for $W_\pi(\lambda)$, it is readily checked via (4.19) and Lemma 4.1 that both $W(\lambda)$ and $W_\pi(\lambda')$ are well defined (i.e., both $G(\lambda)$ and $G_\pi(\lambda')$ are invertible) under the stated assumptions. A straightforward calculation then serves to verify that $W(\lambda)JW_\pi(\lambda')^* = J$, as claimed. ∎

Formula (4.17) is a realization for $W_\pi(\lambda)$, which is centered at the point α'. A realization for $W_\pi(\lambda)$, which is centered at the point α, is readily obtained from (2.13):

$$\begin{aligned} W_\pi(\lambda) &= W_\pi(\alpha) - \delta_\alpha(\lambda) V_\pi G_\pi(\lambda)^{-1} G_\pi(\alpha') G_\pi(\alpha)^{-1} U_\pi \\ &= \widetilde{D}_\pi + \delta_\alpha(\lambda) V_\pi G_\pi(\lambda)^{-1} \widetilde{U}_\pi, \end{aligned} \tag{4.20}$$

with

$$\widetilde{D}_\pi = W_\pi(\alpha) = D_\pi - \delta_{\alpha'}(\alpha) V_\pi G_\pi(\alpha)^{-1} U_\pi$$

and

$$\widetilde{U}_\pi = -G_\pi(\alpha') G_\pi(\alpha)^{-1} U_\pi.$$

4.3. Supplementary Remarks

We remark that formula (4.8) is the upper left-hand corner of the more general formula

$$\begin{bmatrix} A_{\mathrm{R}} & b(\alpha)U_{\mathrm{R}} \\ 0 & I_p \end{bmatrix}^* \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} A_{\mathrm{L}} & b(\alpha)U_{\mathrm{L}} \\ 0 & I_p \end{bmatrix}$$
$$= \begin{bmatrix} B_{\mathrm{R}} & a(\alpha)U_{\mathrm{R}} \\ V_{\mathrm{R}} & D_{\mathrm{R}} \end{bmatrix}^* \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} B_{\mathrm{L}} & a(\alpha)U_{\mathrm{L}} \\ V_{\mathrm{L}} & D_{\mathrm{L}} \end{bmatrix} \tag{4.21}$$

or, in a self-evident notation,

$$\mathcal{A}_{\mathrm{R}}^* \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix} \mathcal{A}_{\mathrm{L}} = \mathcal{B}_{\mathrm{R}}^* \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix} \mathcal{B}_{\mathrm{L}}. \tag{4.22}$$

This formula is equivalent to the functional identity

$$J - W_{\mathrm{R}}(\omega)^* J W_{\mathrm{L}}(\lambda) = \rho_\omega(\lambda) U_{\mathrm{R}}^* G_{\mathrm{R}}(\omega)^{-*} G_{\mathrm{R}}(\alpha)^* P G_{\mathrm{L}}(\alpha) G_{\mathrm{L}}(\lambda)^{-1} U_{\mathrm{L}} \tag{4.23}$$

when P is invertible and the realizations $W_{\mathrm{L}}(\lambda)$ and $W_{\mathrm{R}}(\lambda)$ are both centered at α and minimal.

Similar considerations apply to the formula

$$A_{\mathrm{L}} Q A_{\mathrm{R}}^* - B_{\mathrm{L}} Q B_{\mathrm{R}}^* = U_{\mathrm{L}} J U_{\mathrm{R}}^*, \tag{4.24}$$

which is the upper left-hand corner of the block identity

$$\begin{bmatrix} A_{\mathrm{L}} & 0 \\ b(\alpha)V_{\mathrm{L}} & I_p \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} A_{\mathrm{R}} & 0 \\ b(\alpha)V_{\mathrm{R}} & I_p \end{bmatrix}^*$$
$$= \begin{bmatrix} B_{\mathrm{L}} & U_{\mathrm{L}} \\ a(\alpha)V_{\mathrm{L}} & D_{\mathrm{L}} \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} B_{\mathrm{R}} & U_{\mathrm{R}} \\ a(\alpha)V_{\mathrm{R}} & D_{\mathrm{R}} \end{bmatrix}^*. \tag{4.25}$$

Formula (4.25) is equivalent to the fundamental identity

$$J - W_{\mathrm{L}}(\lambda) J W_{\mathrm{R}}(\omega)^* = \rho_\omega(\lambda) V_{\mathrm{L}} G_{\mathrm{L}}(\lambda)^{-1} G_{\mathrm{L}}(\alpha) Q G_{\mathrm{R}}(\alpha)^* G_{\mathrm{R}}(\omega)^{-*} V_{\mathrm{R}}^* \tag{4.26}$$

when Q is invertible and the realizations for $W_{\mathrm{L}}(\lambda)$ and $W_{\mathrm{R}}(\lambda)$ are both centered at α and minimal. Variations on these two themes are explored in great detail in the next section.

From another point of view, (4.21) and (4.22) are equivalent to the operator identity

$$\left[\mathcal{A}_{\mathrm{L}}\begin{bmatrix} F_{\mathrm{L}}u \\ \xi \end{bmatrix}, \quad \mathcal{A}_{\mathrm{R}}\begin{bmatrix} F_{\mathrm{R}}v \\ \eta \end{bmatrix}\right]_{\mathcal{P}} = \left[\mathcal{B}_{\mathrm{L}}\begin{bmatrix} F_{\mathrm{L}}u \\ \xi \end{bmatrix}, \quad \mathcal{B}_{\mathrm{R}}\begin{bmatrix} F_{\mathrm{R}}v \\ \eta \end{bmatrix}\right]_{\mathcal{P}}, \tag{4.27}$$

wherein

$$\mathcal{A} = \begin{bmatrix} \boldsymbol{A} & b(\alpha)\boldsymbol{U} \\ 0 & I_p \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} \boldsymbol{B} & a(\alpha)\boldsymbol{U} \\ \boldsymbol{V} & \boldsymbol{D} \end{bmatrix}$$

are defined in terms of the operators $\boldsymbol{A}, \ldots, \boldsymbol{D}$, which are specified in the formulas (2.17)–(2.21), and $\mathcal{A}_{\mathrm{L}}$, $\mathcal{B}_{\mathrm{L}}$ and $\mathcal{A}_{\mathrm{R}}$, $\mathcal{B}_{\mathrm{R}}$ denote the corresponding operators based on the minimal realizations $W_{\mathrm{L}}(\lambda)$ and $W_{\mathrm{R}}(\lambda)$ (both of which are centered at α), respectively; i.e., the same formulas hold providing that the appropriate subscripts are added. Here the sesquilinear form in (4.27) is defined on the pair space $\mathcal{P} = (\mathcal{M}_{\mathrm{L}} \oplus \mathbb{C}^p) \times (\mathcal{M}_{\mathrm{R}} \oplus \mathbb{C}^p)$ by the rule

$$\left[\begin{bmatrix} F_{\mathrm{L}}u \\ \xi \end{bmatrix}, \begin{bmatrix} F_{\mathrm{R}}v \\ \eta \end{bmatrix}\right]_{\mathcal{P}} = v^* P u + \eta^* J \xi. \tag{4.28}$$

This rule is meaningful because in terms of the given minimal realization

$$W_{\mathrm{L}}(\lambda) = D_{\mathrm{L}} + \delta_\alpha(\lambda) V_{\mathrm{L}} G_{\mathrm{L}}(\lambda)^{-1} U_{\mathrm{L}},$$

the space

$$\mathcal{M}_{\mathrm{L}} = \operatorname{span}\{F_{\mathrm{L}}(\lambda)u : u \in \mathbb{C}^n\}$$

wherein $F_{\mathrm{L}}(\lambda) = V_{\mathrm{L}} G_{\mathrm{L}}(\lambda)^{-1}$, and a comparable set of formulas hold for $W_{\mathrm{R}}(\lambda)$ and $\mathcal{M}_{\mathrm{R}}$. The main calculations are

$$\begin{aligned}
&\boldsymbol{A}_{\mathrm{L}}\colon F_{\mathrm{L}}(\lambda)u \to F_{\mathrm{L}}(\lambda) A_{\mathrm{L}} G_{\mathrm{L}}(\alpha)^{-1} u, \\
&\boldsymbol{B}_{\mathrm{L}}\colon F_{\mathrm{L}}(\lambda)u \to F_{\mathrm{L}}(\lambda) B_{\mathrm{L}} G_{\mathrm{L}}(\alpha)^{-1} u, \\
&\boldsymbol{U}_{\mathrm{L}}\colon \xi \in \mathbb{C}^p \to F_{\mathrm{L}}(\lambda) U_{\mathrm{L}} \xi, \\
&\boldsymbol{V}_{\mathrm{L}}\colon F_{\mathrm{L}}(\lambda)u \to F_{\mathrm{L}}(\alpha)u,
\end{aligned}$$

and

$$\boldsymbol{D}_{\mathrm{L}}\colon \xi \to D_{\mathrm{L}}\xi,$$

plus of course the corresponding set with sub L replaced by sub R.

5. ISOMETRIES AND COISOMETRIES

In this section we study $p \times q$ matrix-valued functions

$$W(\lambda) = D + \delta_\alpha(\lambda) V G(\lambda)^{-1} U,$$

which are either isometric, $W^* \Sigma_p W = \Sigma_q$, or coisometric, $W \Sigma_q W^* = \Sigma_p$, with respect to the pair of signature matrices Σ_p and Σ_q, on Ω_0. We also consider the case in which $p = q$ and $\Sigma_p = \Sigma_q = J$, and hence W is J unitary on Ω_0.

To avoid repetition it is convenient at this point to summarize a number of the basic assumptions that will be invoked in whole or in part in the sequel.

(A1) $W(\lambda)$ is a $p \times q$ matrix-valued function that admits a realization of the form (2.8) in which $G(\lambda) = a(\lambda)A - b(\lambda)B$ is an $n \times n$ matrix-valued function that is invertible at the point $\alpha \in \Omega$. (It is not assumed here that this realization is minimal.)

(A2) The point α referred to in (A1) admits a reflection $\alpha' \in \Omega_W$ such that

$$a(\alpha)a(\alpha')^* = b(\alpha)b(\alpha')^* \neq 0.$$

(A3) The realization (2.8) is minimal, i.e., $\deg_\rho W = n$.

(A4) $W(\lambda)$ is square, i.e., $p = q$.

If $\det G(\lambda) \not\equiv 0$, then there always exists a point $\alpha \in \Omega$ such that $\det G(\alpha) \neq 0$ and (A2) holds; see the discussion surrounding (1.2).

5.1. *Isometries*

Let

$$H(\lambda) = b(\lambda)A^* - a(\lambda)B^* \tag{5.1}$$

and let

$$\mathcal{A} = \begin{bmatrix} A & b(\alpha)U \\ 0 & I_q \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} B & a(\alpha)U \\ V & D \end{bmatrix} \tag{5.2}$$

$$\widetilde{\mathcal{A}} = \begin{bmatrix} A & 0 \\ b(\alpha)V & I_p \end{bmatrix}, \qquad \widetilde{\mathcal{B}} = \begin{bmatrix} B & U \\ a(\alpha)V & D \end{bmatrix} \tag{5.3}$$

be defined in terms of the entries in (2.8). Operators of the form (5.2) and (5.3) were introduced in [1] to study the colligations associated with a class of infinite-dimensional reproducing kernel Hilbert spaces.

LEMMA 5.1. *If W satisfies assumption* (A1), *then*

$$a(\lambda')^* H(\lambda) = b(\lambda)G(\lambda')^*, \tag{5.4}$$

$$b(\lambda')^* H(\lambda) = a(\lambda)G(\lambda')^*, \tag{5.5}$$

$$\delta_\alpha(\lambda')^* H(\lambda) = \rho_\alpha(\lambda)G(\lambda')^*, \tag{5.6}$$

and

$$W^{\#}(\lambda) = D^* + \rho_\alpha(\lambda)U^* H(\lambda)^{-1} V^* \tag{5.7}$$

at every point $\lambda \in \Omega$ at which the indicated inverse exists ((5.6) *guarantees that* $\det H(\lambda) \not\equiv 0$).

Proof. The first two formulas are easily verified and lead easily to the third. The final formula now drops out from the definition of $W^{\#}$,

$$W^{\#}(\lambda) = W(\lambda')^* = D^* + \delta_\alpha(\lambda')^* U^* G(\lambda')^{-*} V^*,$$

and (5.6). ∎

It is convenient to let

$$\widetilde{\Omega} = \{\lambda \in \Omega \text{ which admit at least one reflection } \lambda' \in \Omega\},$$

and correspondingly to let

$$(\Omega_W)^{\widetilde{}} = \{\lambda \in \Omega_W \text{ which admit at least one reflection } \lambda' \in \Omega_W\}.$$

Clearly $\widetilde{\Omega} \supset \Omega_0$ and is closed under reflections, as is $(\Omega_W)^{\widetilde{}}$. In the sequel we shall deal extensively with $p \times p$ matrix-valued functions that are J unitary in the sense that

$$W(\lambda)JW(\lambda')^* = J$$

for every point $\lambda \in (\Omega_W)^{\widetilde{}}$ (or equivalently for every point $\lambda \in \Omega_W \cap \Omega_0$). Note that this forces $W(\lambda')$ to be the same for all reflections λ' of λ.

LEMMA 5.2. *If W satisfies assumption* (A1), *and if*

$$\mathcal{A}^* \begin{bmatrix} P & 0 \\ 0 & \Sigma_q \end{bmatrix} \mathcal{A} = \mathcal{B}^* \begin{bmatrix} P & 0 \\ 0 & \Sigma_p \end{bmatrix} \mathcal{B} \tag{5.8}$$

for some choice of $P \in \mathbb{C}^{n\times n}$, $\Sigma_p \in \mathbb{C}^{p\times p}$, and $\Sigma_q \in \mathbb{C}^{q\times q}$, then

$$\Sigma_q - W(\omega)^* \Sigma_p W(\lambda) = \rho_\omega(\lambda) U^* G(\omega)^{-*} G(\alpha)^* P G(\alpha) G(\lambda)^{-1} U \tag{5.9}$$

for every pair of points λ and ω at which the indicated inverses exist. In particular

$$W^{\#}(\lambda)\Sigma_p W(\lambda) = \Sigma_q \tag{5.10}$$

for every point $\lambda \in \widetilde{\Omega}$ at which $G(\lambda)$ and $G(\lambda')$ are both invertible. Moreover, if $p = q$ and if Σ_q (and so too Σ_p) is invertible and the triple $\{A, B, U\}$ is (a, b) controllable, then

$$\dim \mathcal{R}_{\mathrm{r}}(W) = \operatorname{rank} P. \tag{5.11}$$

If the realization for W is minimal, then P is invertible.

Proof. To begin with it is readily checked that (5.8) is equivalent to the four identities

$$\begin{aligned} A^*PA - B^*PB &= V^*\Sigma_p V, && (5.12)\\ U^*PH(\alpha)^* &= D^*\Sigma_p V, && (5.13)\\ H(\alpha)PU &= V^*\Sigma_p D, && (5.14)\\ \rho_\alpha(\alpha)U^*PU + D^*\Sigma_p D &= \Sigma_q. && (5.15) \end{aligned}$$

Next, with the aid of (5.12)–(5.15) it is readily seen that

$$\begin{aligned} \Sigma_q - W(\omega)^*\Sigma_p W(\lambda) = U^*G(\omega)^{-*}\{&\rho_\alpha(\alpha)G(\omega)^*PG(\lambda)\\ &- \delta_\alpha(\omega)^*H(\alpha)PG(\lambda) - \delta_\alpha(\lambda)G(\omega)^*PH(\alpha)^*\\ &- \delta_\alpha(\omega)^*\delta_\alpha(\lambda)V^*\Sigma_p V\}G(\lambda)^{-1}U \end{aligned}$$

for every pair of points λ and ω at which the indicated inverses exist and, hence, upon reexpressing G and H in terms of A and B and invoking (5.12), that the term inside the curly brackets

$$\{\cdots\} = \rho_\omega(\lambda)G(\alpha)^*PG(\alpha),$$

by a lengthy but straightforward calculation. Formulas (5.9) and (5.10) are now clear. Finally, since

$$b(\omega)\rho_{\omega'}(\lambda) = -a(\omega')^*\delta_\omega(\lambda) \tag{5.16}$$

for every point $\omega \in \widetilde{\Omega}$, it follows from (5.9) and (5.10) that

$$b(\omega)W(\omega')^*\Sigma_p \frac{W(\lambda) - W(\omega)}{\delta_\omega(\lambda)} = a(\omega')^*U^*G(\omega')^{-*}G(\alpha)^*PG(\alpha)G(\lambda)^{-1}U.$$

Thus if $p = q$ and Σ_q is invertible, then so is Σ_p by (5.10) (as are $W(\omega)$ and $W(\omega')$ whenever $G(\omega)$ and $G(\omega')$ are invertible), and hence

$$\mathcal{R}_{\rm r}(W) = \text{span}\{\xi^* a(\omega')^* U^* G(\omega')^{-*} G(\alpha)^* P G(\alpha) G(\lambda)^{-1} U\},$$

where ξ runs over $\mathbb{C}^n$ and ω' runs over those points in $\widetilde{\Omega}$ at which $G(\omega')$ is invertible. Therefore, by the presumed controllability and Lemma 3.2,

$$\mathcal{R}_{\rm r}(W) = \text{span}\{x^* P G(\alpha) G(\lambda)^{-1} U : x \in \mathbb{C}^n\}.$$

This serves to establish (5.11), since the rows of $G(\lambda)^{-1}U$ are linearly independent by the presumed controllability. Finally, if the realization for W is minimal, then $\dim \mathcal{R}_{\rm r}(W) = n$ and so (5.11) forces P to be invertible. ∎

Much the same sort of analysis leads to the following auxiliary conclusion, which is therefore stated without proof.

LEMMA 5.3. *If W satisfies assumption* (A1), *and if*

$$\widetilde{\mathcal{A}} \begin{bmatrix} Q & 0 \\ 0 & \Sigma_p \end{bmatrix} \widetilde{\mathcal{A}}^* = \widetilde{\mathcal{B}} \begin{bmatrix} Q & 0 \\ 0 & \Sigma_q \end{bmatrix} \widetilde{\mathcal{B}}^* \tag{5.17}$$

for some choice of $Q \in \mathbb{C}^{n\times n}$, $\Sigma_p \in \mathbb{C}^{p\times p}$, and $\Sigma_q \in \mathbb{C}^{q\times q}$, then

$$\Sigma_p - W(\lambda)\Sigma_q W(\omega)^* = \rho_\omega(\lambda) V G(\lambda)^{-1} G(\alpha) Q G(\alpha)^* G(\omega)^{-*} V^* \tag{5.18}$$

for every pair of points λ and ω at which the indicated inverses exist. In particular

$$W(\lambda)\Sigma_q W^{\#}(\lambda) = \Sigma_p \tag{5.19}$$

for every point $\lambda \in \widetilde{\Omega}$ at which $G(\lambda)$ and $G(\lambda')$ are both invertible. Moreover, if $p = q$ and Σ_p (and so too Σ_q) is invertible and the triple $\{V, A, B\}$ is (a, b) observable, then

$$\dim \mathcal{R}_{\rm c}(W) = \operatorname{rank} Q. \tag{5.20}$$

If the realization for W is minimal, then Q is invertible.

Let

$$\mathcal{K}_{\rm r}(W) = \text{closed linear span}\left\{\eta^* \frac{\Sigma_q - W(\omega)^*\Sigma_p W(\lambda)}{\rho_\omega(\lambda)} : \omega \in \Omega_W \text{ and } \eta \in \mathbb{C}^q\right\}$$

and

$$\mathcal{K}_{\mathrm{c}}(W) = \text{closed linear span}\left\{ \frac{\Sigma_p - W(\lambda)\Sigma_q W(\omega)^*}{\rho_\omega(\lambda)}\xi : \omega \in \Omega_W \text{ and } \xi \in \mathbb{C}^p \right\}.$$

If the hypotheses of Lemma 5.2 are met, then by (5.9) and the corollary to Theorem 2.2,

$$\mathcal{K}_{\mathrm{r}}(W) \subseteq \{y^* PG(\alpha)G(\lambda)^{-1}U : y \in \mathbb{C}^n\},$$

with equality if and only if the triple $\{A, B, U\}$ is (a, b) controllable. On the other hand, if the hypotheses of Lemma 5.3 are met, then by (5.18) and the corollary to Theorem 2.2,

$$\mathcal{K}_{\mathrm{c}}(W) \subseteq \{VG(\lambda)^{-1}G(\alpha)Qx : x \in \mathbb{C}^n\}$$

with equality if and only if the triple $\{V, A, B\}$ is (a, b) observable.

If equality prevails in both of these inclusions, then the realization (2.8) is minimal. Moreover,

$$\begin{gathered} \mathcal{K}_{\mathrm{r}}(W) \subseteq \{y^* G(\lambda)^{-1}U : y \in \mathbb{C}^n\} = \mathcal{R}_{\mathrm{r}}(W), \\ \mathcal{K}_{\mathrm{c}}(W) \subseteq \{VG(\lambda)^{-1}x : x \in \mathbb{C}^n\} = \mathcal{R}_{\mathrm{c}}(W), \\ \dim \mathcal{K}_{\mathrm{r}}(W) = \operatorname{rank} P, \qquad \dim \mathcal{K}_{\mathrm{c}}(W) = \operatorname{rank} Q, \end{gathered}$$

and

$$\dim \mathcal{R}_{\mathrm{r}}(W) = \dim \mathcal{K}_{\mathrm{c}}(W) = n.$$

One or more of these last two inclusions may be proper.

The following example, which is adapted from [8], is instructive. Let

$$W(\lambda) = [1 \quad \lambda^3 \quad \lambda^2], \qquad a(\lambda) = 1, \qquad b(\lambda) = \lambda,$$

and fix a point $\alpha \in \mathbb{C}$. Then it is readily checked that $W(\lambda)$ admits a realization of the form (2.8) with

$$D = [1 \quad \alpha^3 \quad \alpha^2], \qquad V = [1 \quad 0 \quad 0],$$

$$A = \begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & -\alpha \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad U = \begin{bmatrix} 0 & \alpha^2 & 0 \\ 0 & -\alpha & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Thus $G(\alpha)$ is clearly invertible for every choice of $\alpha \in \mathbb{C}$. This realization is minimal since $\mu_r = \mu_c = 3$. Moreover, upon setting

$$\Sigma_p = 1 \qquad \text{and} \qquad \Sigma_q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

it is easily seen that (5.17) is uniquely solved by the choice

$$Q = \begin{bmatrix} |\alpha|^4 & 0 & \alpha^2 \\ 0 & 0 & 0 \\ \alpha^{*2} & 0 & 1 \end{bmatrix}.$$

A lengthy but straightforward calculation then serves to verify that (5.18) and (5.19) hold, as they should.

Next, since

$$\frac{\Sigma_p - W(\lambda)\Sigma_q W(\omega)^*}{\rho_\omega(\lambda)} = \lambda^2\omega^{*2},$$

it follows that

$$\mathcal{K}_c(W) = \{\lambda^2 x : x \in \mathbb{C}\}.$$

The same conclusion is obtained from (5.18): since the triple $\{V, A, B\}$ is observable, (5.18) implies that

$$\mathcal{K}_c(W) = \{VG(\lambda)^{-1}G(\alpha)Qx : x \in \mathbb{C}^3\},$$

and hence, as

$$VG(\lambda)^{-1}G(\alpha)Q = \lambda^2[\alpha^{*2} \quad 0 \quad 1],$$

that

$$\begin{aligned}\mathcal{K}_c(W) &= \{\lambda^2[\alpha^{*2} \quad 0 \quad 1]x : x \in \mathbb{C}^3\} \\ &= \{\lambda^2 x : x \in \mathbb{C}\}.\end{aligned}$$

Thus $\mathcal{K}_c(W)$ is a one-dimensional space that is not $r(a, b; \alpha)$ invariant. This (perhaps) somewhat surprising conclusion stems from the fact that $\mu_-(\Sigma_p) \neq \mu_-(\Sigma_q)$.

THEOREM 5.1. *Let $\Sigma_p \in \mathbb{C}^{p\times p}$ and $\Sigma_q \in \mathbb{C}^{q\times q}$ be a pair of signature matrices such that $\mu_-(\Sigma_p) = \mu_-(\Sigma_q)$, and suppose that the $p \times q$ matrix-valued function $W(\lambda)$ satisfies assumption* (A1), *that* $\dim\mathcal{R}_c(W) < \infty$, *and* *that*

$$W(\lambda)\Sigma_q W(\lambda)^* = \Sigma_p$$

for every point $\lambda \in \Omega_W \cap \Omega_0$. *Then*

$$\mathcal{K}_c(W) = \mathcal{R}_c(W).$$

Proof. We may assume without loss of generality that

$$\Sigma_p = \begin{bmatrix} I_s & 0 \\ 0 & -I_k \end{bmatrix} \quad \text{and} \quad \Sigma_q = \begin{bmatrix} I_t & 0 \\ 0 & -I_k \end{bmatrix}.$$

Then, since $\mathcal{K}_c(W) \subseteq \mathcal{R}_c(W)$ and dim $\mathcal{R}_c(W) < \infty$, the kernel

$$K_\omega^W(\lambda) = \frac{\Sigma_p - W(\lambda)\Sigma_q W(\omega)^*}{\rho_\omega(\lambda)}$$

has at most finitely many negative squares. Therefore, if $W(\lambda)$ is partitioned into the block form

$$W(\lambda) = \begin{bmatrix} W_{11}(\lambda) & W_{12}(\lambda) \\ W_{21}(\lambda) & W_{22}(\lambda) \end{bmatrix}$$

with $W_{11}(\lambda) \in \mathbb{C}^{s\times t}$ and $W_{22}(\lambda) \in \mathbb{C}^{k\times k}$, it follows from (1) in Theorem 6.8 of [2], that $\det W_{22}(\lambda) \not\equiv 0$. Thus the Potapov–Ginzburg transform

$$\sigma(\lambda) = \begin{bmatrix} W_{11} - W_{12}W_{22}^{-1}W_{21} & W_{12}W_{22}^{-1} \\ -W_{22}^{-1}W_{21} & W_{22}^{-1} \end{bmatrix}(\lambda)$$

is well defined. The supplementary formulas

$$\frac{\Sigma_p - W(\lambda)\Sigma_q W(\omega)^*}{\rho_\omega(\lambda)} = Z(\lambda)\frac{I_p - \sigma(\lambda)\sigma(\omega)^*}{\rho_\omega(\lambda)}Z(\omega)^*$$

and

$$\frac{W(\lambda) - W(\omega)}{\delta_\omega(\lambda)} = Z(\lambda)\frac{\sigma(\lambda) - \sigma(\omega)}{\delta_\omega(\lambda)}\begin{bmatrix} I_t & 0 \\ W_{21}(\omega) & W_{22}(\omega) \end{bmatrix},$$

wherein

$$Z(\lambda) = \begin{bmatrix} I_s & -W_{12}(\lambda) \\ 0 & W_{22}(\lambda) \end{bmatrix},$$

are easily verified by straightforward calculation. Since $\det W_{22}(\lambda) \not\equiv 0$, these identities serve to identify

$$\mathcal{K}_c(W) = Z(\lambda)\mathcal{K}_c(\sigma)$$

and

$$\mathcal{R}_c(W) = Z(\lambda)\mathcal{R}_c(\sigma),$$

where (in a self-evident extension of the notation) we have set

$$\mathcal{K}_c(\sigma) = \operatorname{span}\left\{\frac{I_p - \sigma(\lambda)I_q\sigma(\omega)^*}{\rho_\omega(\lambda)}\xi : \omega \in \Omega_\sigma \text{ and } \xi \in \mathbb{C}^p\right\}$$

and

$$\mathcal{R}_c(\sigma) = \operatorname{span}\left\{\frac{\sigma(\lambda) - \sigma(\omega)}{\delta_\omega(\lambda)}\eta : \omega \in \Omega_\sigma, |a(\omega)| + |b(\omega)| > 0 \text{ and } \eta \in \mathbb{C}^q\right\}.$$

Since $\mu_-(I_p) = \mu_-(I_q) = 0$, it follows from the corollary to Theorem 6.5 in [2] that

$$\mathcal{R}_c(\sigma) \subseteq \mathcal{K}_c(\sigma).$$

Therefore

$$\mathcal{R}_c(W) = Z(\lambda)\mathcal{R}_c(\sigma) \subseteq Z(\lambda)\mathcal{K}(\sigma) = \mathcal{K}_c(W),$$

which serves to establish equality, since the opposite inclusion is already known. ■

5.2. J Unitary W

THEOREM 5.2. *Let $W(\lambda)$ be a matrix-valued function that satisfies assumptions* (A1)–(A4) *for some point $\alpha \in (\Omega_W)^\sim$ and also meets the identity*

$$W^\#(\lambda)JW(\lambda) = J \tag{5.21}$$

at every point $\lambda \in (\Omega_W)^\sim$ for some $p \times p$ signature matrix J. Then there exists a pair of $n \times n$ invertible matrices P and Q such that

$$\mathcal{A}^*\begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix}\mathcal{A} = \mathcal{B}^*\begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix}\mathcal{B}, \tag{5.22}$$

$$\widetilde{\mathcal{A}}\begin{bmatrix} Q & 0 \\ 0 & J \end{bmatrix}\widetilde{\mathcal{A}}^* = \widetilde{\mathcal{B}}\begin{bmatrix} Q & 0 \\ 0 & J \end{bmatrix}\widetilde{\mathcal{B}}^*, \tag{5.23}$$

and

$$G(\alpha)QG(\alpha)^*P = G(\alpha)^*PG(\alpha)Q = I_n. \tag{5.24}$$

Theorem 5.2 is a special case of the following more general theorem, which can be proved with essentially no extra effort.

THEOREM 5.3. *Let $W_{\mathrm{L}}(\lambda)$ and $W_{\mathrm{R}}(\lambda)$ be a pair of matrix-valued functions that satisfy assumptions* (A1)–(A4) *at a common point $\alpha \in \Omega$, and meet the identity*

$$W_{\mathrm{L}}(\lambda)JW_{\mathrm{R}}^{\#}(\lambda) = J \tag{5.25}$$

for every point $\lambda \in (\Omega_{W_{\mathrm{L}}})^{\sim} \cap (\Omega_{W_{\mathrm{R}}})^{\sim}$ for some $p \times p$ signature matrix J. Then there exists a pair of $n \times n$ invertible matrices P and Q such that

$$\mathcal{A}_{\mathrm{R}}^* \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix} \mathcal{A}_{\mathrm{L}} = \mathcal{B}_{\mathrm{R}}^* \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix} \mathcal{B}_{\mathrm{L}}, \tag{5.26}$$

$$\widetilde{\mathcal{A}}_{\mathrm{L}} \begin{bmatrix} Q & 0 \\ 0 & J \end{bmatrix} \widetilde{\mathcal{A}}_{\mathrm{R}}^* = \widetilde{\mathcal{B}}_{\mathrm{L}} \begin{bmatrix} Q & 0 \\ 0 & J \end{bmatrix} \widetilde{\mathcal{B}}_{\mathrm{R}}^*, \tag{5.27}$$

and

$$G_{\mathrm{L}}(\alpha)QG_{\mathrm{R}}(\alpha)^*P = I_n. \tag{5.28}$$

Proof. Since λ is a reflection of λ', formula (5.25) implies that

$$W_{\mathrm{R}}(\lambda)JW_{\mathrm{L}}^{\#}(\lambda) = J$$

for $\lambda \in (\Omega_{W_{\mathrm{L}}})^{\sim} \cap (\Omega_{W_{\mathrm{R}}})^{\sim}$ and hence that W_{L} may be identified as the partner $(W_{\mathrm{R}})_\pi$ of W_{R}, i.e.,

$$W_{\mathrm{L}}(\lambda) = (W_{\mathrm{R}})_\pi(\lambda).$$

In order to keep the typography simple, we abbreviate $(W_{\mathrm{R}})_\pi(\lambda)$ by $W_\pi(\lambda)$ for the rest of this proof. Because of the last identity, formulas (2.8) for W_{L} and (4.17) for $W_\pi = (W_{\mathrm{R}})_\pi$ yield minimal realizations for the same matrix-valued function. Thus as $G_{\mathrm{L}}(\alpha')$ is invertible by Theorem 3.6, formula (2.13) implies that

$$W_{\mathrm{L}}(\lambda) = W_{\mathrm{L}}(\alpha') + \delta_{\alpha'}(\lambda)V_{\mathrm{L}}G_{\mathrm{L}}(\lambda)^{-1}G_{\mathrm{L}}(\alpha)G_{\mathrm{L}}(\alpha')^{-1}U_{\mathrm{L}},$$

whereas by (4.17)

$$W_\pi(\lambda) = D_\pi - \delta_{\alpha'}(\lambda)V_\pi G_\pi(\lambda)^{-1}U_\pi,$$

where the terms in the last formula are spelled out in the statement of Theorem 4.2, applied to $W_{\rm R}$. The matrix $D_{\rm R}$ is invertible because $W_{\rm R}(\alpha)$ is invertible. Clearly

$$W_{\rm L}(\alpha') = D_\pi = JD_{\rm R}^{-*}J,$$

which upon setting $\lambda = \alpha'$ in (2.8) implies that

$$D_{\rm R}^* JD_{\rm L} + \delta_\alpha(\alpha')D_{\rm R}^* JV_{\rm L}G_{\rm L}(\alpha')^{-1}U_{\rm L} = J. \tag{5.29}$$

Moreover, by Theorem 3.3 there exist a unique pair of invertible matrices X and Y such that

$$\begin{aligned} V_{\rm L}Y &= V_\pi, \qquad -X^{-1}G_{\rm L}(\alpha)G_{\rm L}(\alpha')^{-1}U_{\rm L} = U_\pi, \\ A_\pi &= X^{-1}A_{\rm L}Y, \qquad B_\pi = X^{-1}B_{\rm L}Y. \end{aligned}$$

Upon substituting for V_π, U_π, A_π, and B_π, one obtains the formulas

$$U_{\rm R}^* = D_{\rm R}^* JV_{\rm L}Y, \tag{5.30}$$

$$V_{\rm R}^* = -X^{-1}G_{\rm L}(\alpha)G_{\rm L}(\alpha')^{-1}U_{\rm L}JD_{\rm R}^*, \tag{5.31}$$

$$XG_\pi(\lambda) = G_{\rm L}(\lambda)Y, \tag{5.32}$$

$$A_\pi = k\{B_{\rm R}^* - a(\alpha)^*V_{\rm R}^*D_{\rm R}^{-*}U_{\rm R}^*\} = X^{-1}A_{\rm L}Y, \tag{5.33}$$

and

$$B_\pi = k\{A_{\rm R}^* - b(\alpha)^*V_{\rm R}^*D_{\rm R}^{-*}U_{\rm R}^*\} = X^{-1}B_{\rm L}Y, \tag{5.34}$$

where

$$k = b(\alpha')/a(\alpha)^*, \tag{5.35}$$

for short. ■

The rest of the proof is divided into steps.

Step 1.

$$G_\pi(\lambda) = -k\{H_{\rm R}(\lambda) + \rho_\alpha(\lambda)V_{\rm R}^*D_{\rm R}^{-*}U_{\rm R}^*\} \tag{5.36}$$

for every point $\lambda \in \Omega$.

Proof of Step 1. By definition,

$$\begin{aligned} G_\pi(\lambda) &= a(\lambda)A_\pi - b(\lambda)B_\pi \\ &= a(\lambda)k\{B_{\rm R}^* - a(\alpha)^*V_{\rm R}^*D_{\rm R}^{-*}U_{\rm R}^*\} \\ &\quad -b(\lambda)k\{A_{\rm R}^* - b(\alpha)^*V_{\rm R}^*D_{\rm R}^{-*}U_{\rm R}^*\} \\ &= k\{a(\lambda)B_{\rm R}^* - b(\lambda)A_{\rm R}^* - \rho_\alpha(\lambda)V_{\rm R}^*D_{\rm R}^{-*}U_{\rm R}^*\} \\ &= -k\{H_{\rm R}(\lambda) + \rho_\alpha(\lambda)V_{\rm R}^*D_{\rm R}^{-*}U_{\rm R}^*\}. \end{aligned}$$

Step 2. $-k^2 G_{\mathrm{R}}(\alpha)^* G_\pi(\alpha')^{-1} = I_n$.

Proof of Step 2. By definition,

$$\begin{aligned}
-k^2 G_{\mathrm{R}}(\alpha)^* &= -k\frac{b(\alpha')}{a(\alpha)^*}\left\{a(\alpha)^* A_{\mathrm{R}}^* - b(\alpha)^* B_{\mathrm{R}}^*\right\} \\
&= -k\{b(\alpha')A_{\mathrm{R}}^* - a(\alpha')B_{\mathrm{R}}^*\} \\
&= -kH_{\mathrm{R}}(\alpha') \\
&= G_\pi(\alpha'),
\end{aligned}$$

by Step 1.

Step 3.

$$A_{\mathrm{R}}^* G_\pi(\alpha')^{-1} B_{\mathrm{R}}^* = B_{\mathrm{R}}^* G_\pi(\alpha')^{-1} A_{\mathrm{R}}^*. \tag{5.37}$$

Proof of Step 3. This is established in just the same way as Lemma 2.4, after first expressing G_π in terms of H_{R}, as in Step 1.

Step 4.

$$b(\alpha')\rho_\alpha(\alpha) = \delta_\alpha(\alpha')a(\alpha)^*. \tag{5.38}$$

Proof of Step 4. This is an elementary computation. (It is also a special case of (5.16).)

Step 5. Formulas (5.29), (5.30), and (5.32) imply that

$$D_{\mathrm{R}}^* J D_{\mathrm{L}} + \rho_\alpha(\alpha) U_{\mathrm{R}}^* P U_{\mathrm{L}} = J \tag{5.39}$$

with

$$P = kG_\pi(\alpha')^{-1}X^{-1} = kY^{-1}G_{\mathrm{L}}(\alpha')^{-1}. \tag{5.40}$$

Proof of Step 5. Substitute (5.30) into (5.29) to obtain

$$D_{\mathrm{R}}^* J D_{\mathrm{L}} + \delta_\alpha(\alpha') U_{\mathrm{R}}^* Y^{-1} G_{\mathrm{L}}(\alpha')^{-1} U_{\mathrm{L}} = J.$$

By (5.40) (which rests on (5.32)) and Step 4, the second term on the left is equal to

$$\delta_\alpha(\alpha')k^{-1}U_{\mathrm{R}}^* P U_{\mathrm{L}} = \rho_\alpha(\alpha) U_{\mathrm{R}}^* P U_{\mathrm{L}}.$$

Step 6. Formulas (5.30) and (5.32) imply that

$$D_{\mathrm{R}}^* J V_{\mathrm{L}} = U_{\mathrm{R}}^* P H_{\mathrm{L}}(\alpha)^*. \tag{5.41}$$

Proof of Step 6. By (5.30) and (5.32),

$$\begin{aligned} D_{\mathrm{R}}^* J V_{\mathrm{L}} &= U_{\mathrm{R}}^* Y^{-1} \\ &= U_{\mathrm{R}}^* G_\pi(\alpha')^{-1} X^{-1} G_{\mathrm{L}}(\alpha') \\ &= k^{-1} U_{\mathrm{R}}^* P\{a(\alpha')A_{\mathrm{L}} - b(\alpha')B_{\mathrm{L}}\}, \end{aligned}$$

which is readily seen to reduce to (5.41), since $\rho_\alpha(\alpha') = 0$.

Step 7. Formulas (5.31) and (5.32) and Step 5 imply that

$$V_{\mathrm{R}}^* J D_{\mathrm{L}} = H_{\mathrm{R}}(\alpha) P U_{\mathrm{L}}. \tag{5.42}$$

Proof of Step 7. By (5.31) and (5.32),

$$\begin{aligned} V_{\mathrm{R}}^* &= -X^{-1} G_{\mathrm{L}}(\alpha) G_{\mathrm{L}}(\alpha')^{-1} U_{\mathrm{L}} J D_{\mathrm{R}}^* \\ &= -G_\pi(\alpha) Y^{-1} G_{\mathrm{L}}(\alpha')^{-1} U_{\mathrm{L}} J D_{\mathrm{R}}^* \\ &= -k^{-1} G_\pi(\alpha) P U_{\mathrm{L}} J D_{\mathrm{R}}^*. \end{aligned}$$

Next, by successive applications of Steps 1 and 5, the last identity can be reexpressed as

$$\begin{aligned} V_{\mathrm{R}}^* &= H_{\mathrm{R}}(\alpha) P U_{\mathrm{L}} J D_{\mathrm{R}}^* + \rho_\alpha(\alpha) V_{\mathrm{R}}^* D_{\mathrm{R}}^{-*} U_{\mathrm{R}}^* P U_{\mathrm{L}} J D_{\mathrm{R}}^* \\ &= H_{\mathrm{R}}(\alpha) P U_{\mathrm{L}} J D_{\mathrm{R}}^* + V_{\mathrm{R}}^* D_{\mathrm{R}}^{-*}\{J - D_{\mathrm{R}}^* J D_{\mathrm{L}}\} J D_{\mathrm{R}}^*, \end{aligned}$$

which is equivalent to (5.42).

Step 8. Formulas (5.30), (5.33), and (5.34) imply that

$$A_{\mathrm{R}}^* P A_{\mathrm{L}} - B_{\mathrm{R}}^* P B_{\mathrm{L}} = V_{\mathrm{R}}^* J V_{\mathrm{L}}. \tag{5.43}$$

Proof of Step 8. By (5.33) and (5.34),

$$\begin{aligned} & k A_{\mathrm{R}}^* G_\pi(\alpha')^{-1} A_\pi - k B_{\mathrm{R}}^* G_\pi(\alpha')^{-1} B_\pi \\ &= k A_{\mathrm{R}}^* G_\pi(\alpha')^{-1} X^{-1} A_{\mathrm{L}} Y - k B_{\mathrm{R}}^* G_\pi(\alpha')^{-1} X^{-1} B_{\mathrm{L}} Y \\ &= \{A_{\mathrm{R}}^* P A_{\mathrm{L}} - B_{\mathrm{R}}^* P B_{\mathrm{L}}\} Y. \end{aligned}$$

At the same time, with the help of Steps 3 and 2, the left-hand side of the last formula is readily seen to reduce to

$$-k^2 G_{\mathrm{R}}(\alpha)^* G_\pi(\alpha')^{-1} V_{\mathrm{R}}^* D_{\mathrm{R}}^{-*} U_{\mathrm{R}}^* = V_{\mathrm{R}}^* D_{\mathrm{R}}^{-*} U_{\mathrm{R}}^*.$$

Therefore

$$A_{\mathrm{R}}^* P A_{\mathrm{L}} - B_{\mathrm{R}}^* P B_{\mathrm{L}} = V_{\mathrm{R}}^* D_{\mathrm{R}}^{-*} U_{\mathrm{R}}^* Y^{-1},$$

which yields the desired result via (5.30).

Step 9 completes the proof.

Proof of Step 9. Formulas (5.39), (5.41), (5.42), and (5.43) are clearly equivalent to (5.26).

The identity (5.27) with

$$Q = -kG_{\mathrm{L}}(\alpha)^{-1}X = -kYG_{\pi}(\alpha)^{-1} \tag{5.44}$$

is verified in much the same way. In particular, it follows from the formula $D_{\pi} = W_{\mathrm{L}}(\alpha')$ that

$$J = D_{\mathrm{L}}JD_{\mathrm{R}}^{*} + \delta_{\alpha}(\alpha')V_{\mathrm{L}}G_{\mathrm{L}}(\alpha')^{-1}U_{\mathrm{L}}JD_{\mathrm{R}}^{*}$$

and hence by (5.31) and (5.38) that

$$J = D_{\mathrm{L}}JD_{\mathrm{R}}^{*} + \rho_{\alpha}(\alpha)V_{\mathrm{L}}QV_{\mathrm{R}}^{*}. \tag{5.45}$$

Next, by (5.31) and the formula

$$G_{\mathrm{L}}(\alpha') = kH_{\mathrm{L}}(\alpha)^{*}, \tag{5.46}$$

it is readily seen that

$$H_{\mathrm{L}}(\alpha)^{*}QV_{\mathrm{R}}^{*} = U_{\mathrm{L}}JD_{\mathrm{R}}^{*}. \tag{5.47}$$

On the other hand, by (5.30) and (5.36),

$$\begin{aligned} U_{\mathrm{R}}^{*} &= -k^{-1}D_{\mathrm{R}}^{*}JV_{\mathrm{L}}QG_{\pi}(\alpha) \\ &= D_{\mathrm{R}}^{*}JV_{\mathrm{L}}QH_{\mathrm{R}}(\alpha) + \rho_{\alpha}(\alpha)D_{\mathrm{R}}^{*}JV_{\mathrm{L}}QV_{\mathrm{R}}^{*}D_{\mathrm{R}}^{-*}U_{\mathrm{R}}^{*}, \end{aligned}$$

which, with the help of (5.45), is readily seen to reduce to

$$D_{\mathrm{L}}JU_{\mathrm{R}}^{*} = V_{\mathrm{L}}QH_{\mathrm{R}}(\alpha). \tag{5.48}$$

Next, since $XA_{\pi} = A_{\mathrm{L}}Y$ and $XB_{\pi} = B_{\mathrm{L}}Y$, it follows from Lemma 2.4 that

$$A_{\mathrm{L}}G_{\mathrm{L}}(\alpha)^{-1}XB_{\pi} = B_{\mathrm{L}}G_{\mathrm{L}}(\alpha)^{-1}XA_{\pi},$$

which, upon invoking the definitions of A_{π}, B_{π}, and Q, reduces to

$$A_{\mathrm{L}}QA_{\mathrm{R}}^{*} - B_{\mathrm{L}}QB_{\mathrm{R}}^{*} = H_{\mathrm{L}}(\alpha)^{*}QV_{\mathrm{R}}^{*}D_{\mathrm{R}}^{-*}U_{\mathrm{R}}^{*}.$$

By (5.47) this can be rewritten as

$$A_{\mathrm{L}}QA_{\mathrm{R}}^{*} - B_{\mathrm{L}}QB_{\mathrm{R}}^{*} = U_{\mathrm{L}}JU_{\mathrm{R}}^{*}. \tag{5.49}$$

It is now a simple matter to check that the four identities (5.45), (5.47)–(5.49) are equivalent to (5.27).

Finally, (5.28) is readily verified via the definitions (5.40) and (5.44) and Step 2. ■

THEOREM 5.4. *If the matrix-valued function $W(\lambda)$ satisfies assumptions* (A1)–(A4) *and if $W(\alpha)$ and $G(\alpha')$ are invertible and J is a $p \times p$ signature matrix, then Eq.* (5.22),

$$\mathcal{A}^* \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix} \mathcal{A} = \mathcal{B}^* \begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix} \mathcal{B},$$

has at most one solution P. It is automatically both invertible and Hermitian.

Proof. In view of Lemma 5.2 and the presumed minimality of the realization, every solution P of Eq. (5.22) is invertible. Therefore we may suppose from the outset that Eq. (5.22) admits an invertible solution P. Then by Lemma 5.2 with $\Sigma_p = \Sigma_q = J$,

$$W(\lambda) J W^{\#}(\lambda) = J$$

for every point $\lambda \in (\Omega_W)^{\sim}$. Therefore $W(\lambda) = W_\pi(\lambda)$ and hence, since the realizations

$$W(\lambda) = W(\alpha') + \delta_{\alpha'}(\lambda) V G(\lambda)^{-1} G(\alpha) G(\alpha')^{-1} U$$

and

$$W_\pi(\lambda) = D_\pi - \delta_{\alpha'}(\lambda) V_\pi G_\pi(\lambda)^{-1} U_\pi$$

are both minimal, Theorem 3.3 guarantees the existence of a unique pair of matrices X and Y such that

$$-XU_\pi = G(\alpha)G(\alpha')^{-1}U, \qquad XA_\pi = AY, \qquad XB_\pi = BY, \qquad VY = V_\pi. \tag{5.50}$$

The strategy of the proof is to show that the choices

$$X = P^{-1} k G_\pi(\alpha')^{-1} \qquad \text{and} \qquad Y = kG(\alpha')^{-1} P^{-1}$$

work. The asserted uniqueness of P then emerges from the uniqueness of X and Y once the point α' is fixed. The tools are formulas (5.39) and (5.41)–(5.43), which are equivalent to (5.22) when $\mathcal{A}_{\mathrm{R}} = \mathcal{A}_{\mathrm{L}} = \mathcal{A}$ and $\mathcal{B}_{\mathrm{R}} = \mathcal{B}_{\mathrm{L}} = \mathcal{B}$. The proof is divided into steps.

Step 1. $VY = V_\pi$.

Proof of Step 1. By definition,

$$\begin{aligned} V_\pi Y^{-1} &= k^{-1} J D^{-*} U^* P G(\alpha') \\ &= J D^{-*} U^* P H(\alpha)^*, \end{aligned}$$

which is clearly equal to V by (5.41).

Step 2. $XA_\pi = AY$ and $XB_\pi = BY$.

Proof of Step 2. By definition and formula (5.46),

$$\begin{aligned} A_\pi Y^{-1} &= \{B^* - a(\alpha)^* V^* D^{-*} U^*\} P G(\alpha') \\ &= B^* P G(\alpha') - a(\alpha)^* k V^* D^{-*} U^* P H(\alpha)^*. \end{aligned}$$

Therefore, by successive applications of (5.41), (5.43), and (5.36),

$$\begin{aligned} A_\pi Y^{-1} &= B^* P G(\alpha') - b(\alpha') V^* J V \\ &= B^* P\{a(\alpha')A - b(\alpha')B\} - b(\alpha')\{A^* P A - B^* P B\} \\ &= -H(\alpha') P A \\ &= k^{-1} G_\pi(\alpha') P A \\ &= X^{-1} A, \end{aligned}$$

as needed.

The proof of the second assertion goes through in exactly the same way.

Step 3.

$$G_\pi(\alpha')^{-1} G_\pi(\lambda) P = P G(\lambda) G(\alpha')^{-1}$$

for every choice of $\lambda \in \Omega$.

Proof of Step 3. By Step 2,

$$X G_\pi(\lambda) = G(\lambda) Y.$$

The rest is immediate upon substituting the chosen recipes for X and Y.

Step 4. $XU_\pi = -G(\alpha) G(\alpha')^{-1} U$.

Proof of Step 4. By definition,

$$-X^{-1}G(\alpha)G(\alpha')^{-1}U = -k^{-1}G_\pi(\alpha')PG(\alpha)G(\alpha')^{-1}U.$$

By Step 3 with $\lambda = \alpha$, followed by successive applications of (5.36), (5.42), and (5.39), the right-hand side of the last formula can be rewritten as

$$\begin{aligned}-k^{-1}G_\pi(\alpha)PU &= H(\alpha)PU + \rho_\alpha(\alpha)V^*D^{-*}U^*PU\\ &= V^*JD + V^*D^{-*}\{J - D^*JD\}\\ &= V^*D^{-*}J\\ &= U_\pi,\end{aligned}$$

as claimed.

Step 5. P is unique and $P = P^*$.

Proof of Step 5. The preceding four steps guarantee that the chosen X and Y (which are defined in terms of any invertible solution P of (5.22)) do indeed satisfy (5.50). Therefore, since there is only one such choice of X and Y by Theorem 3.3, there is at most one invertible solution P of (5.22). Since P^* is also an invertible solution of (5.22) whenever P is, it follows that $P = P^*$. This completes the proof of the step and the theorem. ∎

THEOREM 5.5. *If the matrix-valued function $W(\lambda)$ satisfies assumptions* (A1)–(A4), *and if also $W(\alpha)$ and $G(\alpha')$ are invertible and J is a $p \times p$ signature matrix, then Eq.* (5.23),

$$\widetilde{\mathcal{A}}\begin{bmatrix} Q & 0\\ 0 & J\end{bmatrix}\widetilde{\mathcal{A}}^* = \widetilde{\mathcal{B}}\begin{bmatrix} Q & 0\\ 0 & J\end{bmatrix}\widetilde{\mathcal{B}}^*,$$

has at most one solution Q. It is automatically both invertible and Hermitian.

Proof. In view of Lemma 5.3 and the presumed minimality of the realization, every solution Q of Eq. (5.23) is invertible. Therefore we may suppose from the outset that Eq. (5.23) admits an invertible solution Q. Then by Lemma 5.3 with $\Sigma_p = \Sigma_q = J$,

$$W(\lambda)JW^{\#}(\lambda) = J$$

for every point $\lambda \in (\Omega_W)^{\sim}$. Therefore, just as in the first few lines of the proof of the preceding theorem, there exists a unique pair of matrices X

and Y such that (5.50) holds. The strategy of the rest of the proof is to show that the choice

$$X = -k^{-1}G(\alpha)Q \qquad \text{and} \qquad Y = -k^{-1}QG_\pi(\alpha)$$

works. The verification is divided into steps that go through much as before except that now we use the formulas (5.45) and (5.47)–(5.49) with the subscripts L and R deleted instead of (5.39) and (5.41)–(5.43).

Step 1. $XU_\pi = -G(\alpha)G(\alpha')^{-1}U$.

Proof of Step 1. By definition,

$$-G(\alpha')G(\alpha)^{-1}XU_\pi = H(\alpha)^*QV^*D^{-*}J.$$

The rest is immediate from (5.47).

Step 2. $XA_\pi = AY$ and $XB_\pi = BY$.

Proof of Step 2. By definition

$$\begin{aligned} XA_\pi - AY &= -k^{-1}G(\alpha)QA_\pi + k^{-1}AQG_\pi(\alpha) \\ &= -k^{-1}\{a(\alpha)A - b(\alpha)B\}QA_\pi + k^{-1}AQ\{a(\alpha)A_\pi - b(\alpha)B_\pi\} \\ &= b(\alpha)k^{-1}\{BQA_\pi - AQB_\pi\} \\ &= b(\alpha)BQ\{B^* - a(\alpha)^*V^*D^{-*}U^*\} \\ &\quad - b(\alpha)AQ\{A^* - b(\alpha)^*V^*D^{-*}U^*\} \\ &= b(\alpha)\{BQB^* - AQA^*\} + b(\alpha)H(\alpha)^*QV^*D^{-*}U^*. \end{aligned}$$

But this is clearly equal to zero, thanks to (5.47) and (5.49). This completes the proof of the first assertion. The second goes through in much the same way.

Step 3. $VY = V_\pi$.

Proof of Step 3. By definition and (5.36),

$$\begin{aligned} VY &= -k^{-1}VQG_\pi(\alpha) \\ &= VQH(\alpha) + \rho_\alpha(\alpha)VQV^*D^{-*}U^*. \end{aligned}$$

The rest is immediate from (5.48), (5.45), and the definition of V_π.

Step 4. Q is unique and $Q = Q^*$.

Proof of Step 4. The preceding steps guarantee that the chosen X and Y (which are now defined in terms of any invertible solution Q of (5.23)) do indeed satisfy (5.50). Therefore, since there is only one such choice of X and Y by Theorem 3.3, there is at most one invertible solution Q of (5.23). Since Q^* is also an invertible solution of (5.23) whenever Q is, it follows that $Q = Q^*$. This completes the proof of the step and the theorem. ∎

In the special case that $a(\lambda) = 1$, $b(\lambda) = \lambda$ and $\alpha = 0$, the point $\alpha' = \infty$ does not belong to Ω. Therefore condition (A2) fails. Nevertheless, the conclusions of Theorems 5.2 and 5.4 remain valid for the choice $A = I_n$ and $B = T$ (corresponding to the realization $W(\lambda) = D+\lambda V(I_n-\lambda T)^{-1}U)$ by standard arguments if T is invertible. In this instance

$$\mathcal{A} = \widetilde{\mathcal{A}} = \begin{bmatrix} I_n & 0 \\ 0 & I_p \end{bmatrix} \qquad \text{and} \qquad \mathcal{B} = \widetilde{\mathcal{B}} = \begin{bmatrix} T & U \\ V & D \end{bmatrix}.$$

Thus if say P_1 and P_2 are two solutions of (5.8) with $\Sigma_q = \Sigma_p = J$, it is readily seen that $\Delta = P_1 - P_2$ is a solution of the equations

$$\Delta = T^*\Delta T \qquad \text{and} \qquad 0 = T^*\Delta U.$$

But this in turn implies that

$$\Delta = T^{*k}\Delta T^k \qquad \text{for} \quad k = 0, \ldots, n-1$$

and hence, since T is invertible, that

$$0 = \Delta T^k U \qquad \text{for} \quad k = 0, \ldots, n-1.$$

The presumed minimality of the realization guarantees that the pair $\{T, U\}$ is controllable and hence that $\Delta = 0$. Thus (5.8) has at most one solution. Since P^* is a solution of (5.8) whenever P is, every solution of (5.8) is automatically Hermitian. It is also invertible by Lemma 5.2. This completes the verification of the conclusions of Theorem 5.4 for this case. The verification of the conclusions of Theorem 5.5 is carried out in much the same way, except that now it is the observability of the pair $\{V, T\}$ that comes into play.

The main implications of the preceding analysis can be summarized as follows:

THEOREM 5.6. *If the matrix-valued function $W(\lambda)$ satisfies assumptions* (A1)–(A4) *and if also the matrices $W(\alpha)$ and $G(\alpha')$ are invertible and J is a $p \times p$ signature matrix, then the following are equivalent:*

(1) *$W(\lambda)JW^{\#}(\lambda) = J$ for every point $\lambda \in (\Omega_W)^{\sim}$.*

(2) *There exists a matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\mathcal{A}^*\begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix}\mathcal{A} = \mathcal{B}^*\begin{bmatrix} P & 0 \\ 0 & J \end{bmatrix}\mathcal{B},$$

where $\mathcal{A}$ and $\mathcal{B}$ are defined in terms of the realization (2.8) *of $W(\lambda)$ by* (5.2).

(3) *There exists a matrix $Q \in \mathbb{C}^{n\times n}$ such that*

$$\widetilde{\mathcal{A}}\begin{bmatrix} Q & 0 \\ 0 & J \end{bmatrix}\widetilde{\mathcal{A}}^* = \widetilde{\mathcal{B}}\begin{bmatrix} Q & 0 \\ 0 & J \end{bmatrix}\widetilde{\mathcal{B}}^*,$$

where $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ are defined in terms of the realization (2.8) *of $W(\lambda)$ by* (5.3).

Moreover, the matrices P and Q, which intervene in (2) *and* (3), *are unique. They are also invertible and Hermitian and are linked by formula* (5.24):

$$G(\alpha)QG(\alpha)^* = P^{-1}.$$

We remark that Theorem 3.1 of [13] is a special case of Theorem 5.5, which is obtained by choosing $a(\lambda) = 1$, $b(\lambda) = \lambda$, and $\alpha = 1$. The matrices F and H of that paper correspond to $QH(1) = QG(1)^*$ and P of this one, respectively. Since $G(1)QG(1)^* = P^{-1}$, it follows readily that (3.16) and (3.17) of [13] are special cases of (5.18) and (5.9) in the present paper, respectively. This incorporates the main part of Theorem 3.2 in [13]. The rest follows: The identification of the number of negative eigenvalues of P with the number of negative squares of the kernel $\{J - W(\lambda)JW(\omega)^*\}/\rho_\omega(\lambda)$ and the McMillan degree of W with the dimension of the reproducing kernel space $\mathcal{K}_c(W)$ was obtained earlier in [2]; see especially Theorems 6.12 and 5.7 and also Theorem 2.1 of [3] and Theorem 3.1 of [11].

5.3. Back to the Nonsquare Case

The analysis in the preceding subsection can be partially adapted to the nonsquare case by working with generalized inverses for D. This approach has been systematically developed earlier by Rakowski to study problems of factorization and inversion for a class of realizations (corresponding to the line and to the circle) for rational matrix-valued functions; see [19–21]. We begin with a nonsquare analogue of Theorem 4.2.

THEOREM 5.7. *Suppose that $W(\lambda)$ satisfies assumptions* (A1)–(A3), *that $D = W(\alpha)$ has* rank p, *and that Σ_p and Σ_q are now both signature matrices. Let $D^\dagger$ be a right inverse of D, and let*

$$\begin{aligned}
D_\pi &= \Sigma_p(D^\dagger)^*\Sigma_q, \qquad V_\pi = \Sigma_p(D^\dagger)^*U^*, \qquad U_\pi = V^*(D^\dagger)^*\Sigma_q,\\
A_\pi &= k\{B^* - a(\alpha)^*V^*(D^\dagger)^*U^*\},\\
B_\pi &= k\{A^* - b(\alpha)^*V^*(D^\dagger)^*U^*\},\\
G_\pi(\lambda) &= a(\lambda)A_\pi - b(\lambda)B_\pi,
\end{aligned}$$

and

$$W_\pi(\lambda) = D_\pi - \delta_{\alpha'}(\lambda)V_\pi G_\pi(\lambda)^{-1}U_\pi. \tag{5.51}$$

Then

$$W(\lambda)\Sigma_q W_\pi^\#(\lambda) = \Sigma_p$$

for every point $\lambda \in (\Omega_W)^\sim$ with a reflection $\lambda' \in (\Omega_{W_\pi})^\sim$.

Proof. The verification is by direct calculation, just as in the proof of Theorem 4.2. ■

It is useful to note that

$$W_\pi(\lambda) = \Sigma_p W^\dagger(\lambda')^*\Sigma_q,$$

where

$$W^\dagger(\lambda) = D^\dagger - \delta_\alpha(\lambda)D^\dagger V\{G(\lambda) + \delta_\alpha(\lambda)UD^\dagger V\}^{-1}UD^\dagger \tag{5.52}$$

is a right inverse of $W(\lambda)$:

$$W(\lambda)W^\dagger(\lambda) = I_p. \tag{5.53}$$

THEOREM 5.8. *If $W(\lambda)$ satisfies* (A1) *and* (A2) *for some point $\alpha \in \Omega_0$ (i.e., with $\alpha' = \alpha$) and if the corresponding realization for*

$$\begin{aligned}
W_1(\lambda) &= W(\lambda)\Sigma_q D^*\Sigma_p\\
&= D_1 + \delta_\alpha(\lambda)VG(\lambda)^{-1}U_1,
\end{aligned}$$

with

$$D_1 = D\Sigma_q D^*\Sigma_p \qquad \text{and} \qquad U_1 = U\Sigma_q D^*\Sigma_p,$$

is minimal and Σ_p *and* Σ_q *are both signature matrices, then the following are equivalent:*

(1) *The vector space*

$$\mathrm{row}\{W(\lambda)\} = \textit{the span of the rows of } W(\lambda)$$

is independent of the choice of $\lambda \in \Omega_W$, *and*

$$W(\lambda)\Sigma_q W^{\#}(\lambda) = \Sigma_p \tag{5.54}$$

for every point $\lambda \in (\Omega_W)^{\sim}$.

(2) *The formula*

$$D\Sigma_q D^* = \Sigma_p \tag{5.55}$$

holds, and there exists an invertible Hermitian matrix P *such that*

$$V^*\Sigma_p D = H(\alpha)PU \tag{5.56}$$

and

$$A^*PA - B^*PB = V^*\Sigma_p V. \tag{5.57}$$

(3) *The formulas*

$$D\Sigma_q D^* = \Sigma_p \qquad \textit{and} \qquad U = U_1 D \tag{5.58}$$

hold, and there exists an invertible Hermitian matrix $Q \in \mathbb{C}^{n\times n}$ *such that*

$$\Sigma_p U_1^* = VQH(\alpha) \tag{5.59}$$

and

$$AQA^* - BQB^* = U_1\Sigma_p U_1^*. \tag{5.60}$$

Moreover, if any one (and hence all three) of these conditions are met, then the matrices P *and* Q *which intervene in* (2) *and* (3) *are unique and*

$$G(\alpha)QG(\alpha)^*P = G(\alpha)^*PG(\alpha)Q = I_n, \tag{5.61}$$

$$W(\lambda) = W_j(\lambda)D, \qquad \textit{for} \quad j = 1, 2, \tag{5.62}$$

where

$$W_1(\lambda) = W(\lambda)\Sigma_q D^*\Sigma_p$$

can be expressed in the form

$$W_1(\lambda) = I_p - \rho_\alpha(\lambda)\Sigma_p U_1^* G(\alpha)^{-*}Q^{-1}G(\lambda)^{-1}U_1, \tag{5.63}$$

$$W_2(\lambda) = I_p - \rho_\alpha(\lambda)VG(\lambda)^{-1}P^{-1}G(\alpha)^{-*}V^*\Sigma_p, \tag{5.64}$$

and

$$W_j(\lambda)\Sigma_p W_j^{\#}(\lambda) = W_j^{\#}(\lambda)\Sigma_p W_j(\lambda) = \Sigma_p \tag{5.65}$$

for every point $\lambda \in (\Omega_{W_j})^{\sim}$.

Proof. Suppose first that (1) holds. Then clearly

$$D\Sigma_q D^* = W(\alpha)\Sigma_q W^{\#}(\alpha) = \Sigma_p.$$

This proves (5.55) and hence also guarantees that D is right invertible and $D_1 = I_p$. The auxiliary assumption on the constancy of row$\{W(\lambda)\}$ further guarantees the existence of a $p \times p$ matrix-valued function $E(\lambda)$ that is invertible on Ω_W such that

$$W(\lambda) = E(\lambda)D.$$

In view of (5.55),

$$E(\lambda)\Sigma_p E^{\#}(\lambda) = W(\lambda)\Sigma_q W^{\#}(\lambda) = \Sigma_p$$

and

$$E(\lambda) = W(\lambda)\Sigma_q D^*\Sigma_p = W_1(\lambda),$$

which together clearly imply (5.62) for $j = 1$ and (5.63). Moreover, since $W_1(\lambda) = I_p + \delta_\alpha(\lambda)VG(\lambda)^{-1}U_1$ satisfies assumptions (A1)–(A4), the remaining assertions of both (2) and (3) are immediate from Theorem 5.6 with $J = \Sigma_p$,

$$\mathcal{A} = \begin{bmatrix} A & b(\alpha)U_1 \\ 0 & I_p \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} B & a(\alpha)U_1 \\ V & I_p \end{bmatrix},$$

$$\widetilde{\mathcal{A}} = \begin{bmatrix} A & 0 \\ b(\alpha)V & I_p \end{bmatrix} \quad \text{and} \quad \widetilde{\mathcal{B}} = \begin{bmatrix} B & U_1 \\ a(\alpha)V & I_p \end{bmatrix},$$

with the help of the auxiliary formula

$$U = U_1 D \tag{5.66}$$

(which is justified below). The point is that Theorem 5.6 guarantees the existence of a pair of invertible, Hermitian matrices P and Q such that the formulas (5.57), (5.60),

$$V^*\Sigma_p = H(\alpha)PU_1$$

and

$$VQH(\alpha) = \Sigma_p U_1^* = \Sigma_p(U\Sigma_q D^*\Sigma_p)^* = D\Sigma_q U^*$$

hold. The last formula includes (5.59). The one before it yields

$$V^*\Sigma_p D = H(\alpha)PU_1 D = H(\alpha)PU,$$

by (5.66). This proves (5.56). Formula (5.66) is obtained from the identity

$$W(\lambda) = W_1(\lambda)D$$

by matching the realizations,

$$D + \delta_\alpha(\lambda)VG(\lambda)^{-1}U = D + \delta_\alpha(\lambda)VG(\lambda)^{-1}U_1 D,$$

and invoking the linear independence of the columns of $VG(\lambda)^{-1}$. This completes the proof of (5.58).

Theorem 5.6 also guarantees that P and Q are unique and are linked by (5.61). Moreover, by (5.59),

$$W_1(\lambda) = I_p + \delta_\alpha(\lambda)\Sigma_p U_1^* H(\alpha)^{-1}Q^{-1}G(\lambda)^{-1}U_1,$$

which is readily seen to be the same as (5.63) since $\alpha \in \Omega_0$. Thus to this point we have shown that if (1) holds, then so do assertions (2) and (3), and formulas (5.61), (5.63), (5.62), and (5.65) for $j = 1$ (where applicable).

Now suppose that (2) holds. Then by (5.56),

$$\begin{aligned} W(\lambda) &= D + \delta_\alpha(\lambda)VG(\lambda)^{-1}U \\ &= \{I_p + \delta_\alpha(\lambda)VG(\lambda)^{-1}P^{-1}H(\alpha)^{-1}V^*\Sigma_p\}D, \end{aligned}$$

which is readily seen to reduce to

$$W(\lambda) = W_2(\lambda)D$$

since $\alpha \in \Omega_0$. This serves to guarantee that the vector space row $\{W(\lambda)\}$ is independent of $\lambda \in \Omega_W$ since $\det W_2(\lambda) \not\equiv 0$. Indeed, a routine calculation based on (5.57) yields

$$W_2(\lambda)\Sigma_p W_2^{\#}(\lambda) = W_2^{\#}(\lambda)\Sigma_p W_2(\lambda) = \Sigma_p$$

for every point $\lambda \in (\Omega_{W_2})^{\sim}$. This serves to establish (5.62) and (5.65) for $j = 2$. Also,

$$W(\lambda)\Sigma_q W^{\#}(\lambda) = W_2(\lambda)D\Sigma_q D^* W_2^{\#}(\lambda) = \Sigma_p$$

to complete the proof that (2) $\Rightarrow$ (1).

Finally let us suppose that (3) is in force. Then by (5.58) and (5.59),

$$\begin{aligned} W(\lambda) &= D + \delta_\alpha(\lambda)VG(\lambda)^{-1}U \\ &= D + \delta_\alpha(\lambda)\Sigma_p U_1^* H(\alpha)^{-1}Q^{-1}G(\lambda)^{-1}U_1 D \\ &= W_1(\lambda)D. \end{aligned}$$

This serves to prove that the vector space row $\{W(\lambda)\}$ is independent of λ, whereas by (5.58) and (5.60),

$$W(\lambda)\Sigma_q W^{\#}(\lambda) = W_1(\lambda)\Sigma_p W_1^{\#}(\lambda) = \Sigma_p$$

for every point $\lambda \in (\Omega_W)^{\sim}$. Therefore the proof that (3) $\Rightarrow$ (1) is complete, as is the proof of the theorem. ∎

The next theorem is the counterpart of Theorem 5.8 when D is left invertible and the span of the columns of $W(\lambda)$ is independent of λ. It is stated without proof since the verification is easily adapted from the verification of Theorem 5.7.

THEOREM 5.9. *If $W(\lambda)$ satisfies* (A1) *and* (A2) *for some point $\alpha \in \Omega_0$ (i.e., with $\alpha' = \alpha$) and if the realization for*

$$\begin{aligned} W_3(\lambda) &= \Sigma_q D^* \Sigma_p W(\lambda) \\ &= D_3 + \delta_\alpha(\lambda)V_3 G(\lambda)^{-1}U \end{aligned}$$

with

$$D_3 = \Sigma_q D^* \Sigma_p \qquad \text{and} \qquad V_3 = \Sigma_q D^* \Sigma_p V$$

is minimal and if Σ_p and Σ_q are both signature matrices, then the following are equivalent:

(1) *The vector space*

$$\operatorname{col}\{W(\lambda)\} = \text{the span of the columns of } W(\lambda)$$

is independent of the choice of $\lambda \in \Omega_W$, and

$$W^{\#}(\lambda)\Sigma_p W(\lambda) = \Sigma_q \tag{5.67}$$

for every point $\lambda \in (\Omega_W)^{\sim}$.

(2) *The formulas*

$$D^*\Sigma_p D = \Sigma_q \qquad \text{and} \qquad V = DV_3 \tag{5.68}$$

hold, and there exists an invertible Hermitian matrix $P \in \mathbb{C}^{n\times n}$ *such that*

$$V_3^*\Sigma_q = H(\alpha)PU \tag{5.69}$$

and

$$A^*PA - B^*PB = V_3^*\Sigma_q V_3. \tag{5.70}$$

(3) *The formula*

$$D^*\Sigma_p D = \Sigma_q$$

holds, and there exists an invertible Hermitian matrix $Q \in \mathbb{C}^{n\times n}$ *such that*

$$D\Sigma_q U^* = VQH(\alpha) \tag{5.71}$$

and

$$AQA^* - BQB^* = U\Sigma_q U^*. \tag{5.72}$$

Moreover, if any one (and hence all three) of these conditions are met, then the matrices P *and* Q*, which intervene in* (2) *and* (3)*, are unique and*

$$\begin{aligned} G(\alpha)QG(\alpha)^*P = G(\alpha)^*PG(\alpha)Q &= I_n, \\ W(\lambda) = DW_j(\lambda), \qquad &\text{for} \quad j = 3,4, \end{aligned} \tag{5.73}$$

where

$$W_3(\lambda) = \Sigma_q D^*\Sigma_p W(\lambda)$$

can be expressed in the form

$$W_3(\lambda) = I_q - \rho_\alpha(\lambda)V_3 G(\lambda)^{-1}P^{-1}G(\alpha)^{-*}V_3^*\Sigma_q, \tag{5.74}$$

$$W_4(\lambda) = I_q - \rho_\alpha(\lambda)\Sigma_q U^*G(\alpha)^{-*}Q^{-1}G(\lambda)^{-1}U, \tag{5.75}$$

and

$$W_j(\lambda)\Sigma_q W_j^{\#}(\lambda) = W_j^{\#}(\lambda)\Sigma_q W_j(\lambda) = \Sigma_q \tag{5.76}$$

for every point $\lambda \in (\Omega_{W_j})^{\sim}$ *and* $j = 3,4$.

6. IMBEDDINGS

In this section we show how to imbed the identity (5.12) [resp. the upper left-hand corners of (5.17)] into the "colligation" identity (5.8) [resp.(5.17)] by suitable choice of U [resp. V] and D.

THEOREM 6.1. *Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^{p\times n}$, $\Sigma_p \in \mathbb{C}^{p\times p}$ and $\Sigma_q \in \mathbb{C}^{q\times q}$ be a given set of matrices such that:*

(1) *Σ_p and Σ_q are signature matrices with $p \geq q$ and $\mu_\pm(\Sigma_p) \geq \mu_\pm(\Sigma_q)$.*
(2) *There exists an invertible Hermitian matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$A^*PA - B^*PB = V^*\Sigma_p V. \tag{6.1}$$

(3) *There exists a point $\lambda \in \Omega$ such that $G(\lambda) = a(\lambda)A - b(\lambda)B$ is invertible.*

Then there exist matrices $D_0 \in \mathbb{C}^{p\times q}$ such that

$$D_0^*\Sigma_p D_0 = \Sigma_q \tag{6.2}$$

and points $\beta \in \Omega_0$ such that $G(\beta)$ is invertible. Moreover, for any such choice of D_0 and β, and for every point $\omega \in \Omega$ at which $G(\omega)$ is invertible, the matrices

$$D = \left\{I_p - \rho_\beta(\omega)VG(\omega)^{-1}P^{-1}G(\beta)^{-*}V^*\Sigma_p\right\} D_0 \tag{6.3}$$

and

$$U = G(\beta)G(\omega)^{-1}P^{-1}H(\beta)^{-1}V^*\Sigma_p D_0 \tag{6.4}$$

are solutions of the equations

$$H(\omega)PU = V^*\Sigma_p D \tag{6.5}$$

and

$$\rho_\omega(\omega)U^*PU + D^*\Sigma_p D = \Sigma_q \tag{6.6}$$

(which fill out (5.8)). The corresponding matrix-valued function

$$W(\lambda) = D + \delta_\omega(\lambda)VG(\lambda)^{-1}U$$

can be expressed in the form

$$W(\lambda) = W_0(\lambda)D_0,$$

where

$$\begin{aligned} W_0(\lambda) &= I_p + \delta_\beta(\lambda) V G(\lambda)^{-1} U_0 \\ &= I_p - \rho_\beta(\lambda) V G(\lambda)^{-1} P^{-1} G(\beta)^{-*} V^* \Sigma_p \qquad (6.7) \\ &= I_p - \rho_\beta(\lambda) \Sigma_p U_0^* G(\beta)^{-*} \{G(\beta)^* P G(\beta)\} G(\lambda)^{-1} U_0, \\ U_0 &= P^{-1} H(\beta)^{-1} V^* \Sigma_p, \qquad (6.8) \end{aligned}$$

$$\frac{\Sigma_p - W_0(\lambda)\Sigma_p W_0(\omega)^*}{\rho_\omega(\lambda)} = V G(\lambda)^{-1} P^{-1} G(\omega)^{-*} V^* \qquad (6.9)$$

and

$$\frac{\Sigma_p - W_0(\omega)^*\Sigma_p W_0(\lambda)}{\rho_\omega(\lambda)} = U_0^* G(\omega)^{-*} \{G(\beta)^* P G(\beta)\} G(\lambda)^{-1} U_0. \qquad (6.10)$$

Proof. The existence of a matrix $D_0 \in \mathbb{C}^{p\times q}$, which satisfies (6.2), and a point $\beta \in \Omega_0$ at which $G(\beta)$ (and hence also $H(\beta)$) is invertible is self-evident from the given assumptions. The choices $D = D_0$ and $U = U_0 D_0$ satisfy (6.5) and (6.6) for $\omega = \beta$. Therefore, by Lemma 5.2,

$$W(\lambda) = \{I_p + \delta_\beta(\lambda) V G(\lambda)^{-1} U_0\} D_0$$

satisfies (5.10), $W^\# \Sigma_p W = \Sigma_q$, whereas

$$\begin{aligned} W_0(\lambda) &= I_p + \delta_\beta(\lambda) V G(\lambda)^{-1} U_0 \\ &= I_p - \rho_\beta(\lambda) V G(\lambda)^{-1} P^{-1} G(\beta)^{-*} V^* \Sigma_p \end{aligned}$$

satisfies (6.9). The last claim may be checked by either invoking (5.18) and Theorem 5.6 with $\Sigma_p = \Sigma_q = J$ or by direct calculation (using (6.1)).

The third formula for $W_0(\lambda)$ in (6.7) is easily derived from the first with the aid of the first of the two identities

$$\delta_\beta(\lambda) H(\beta)^* = -\rho_\beta(\lambda) G(\beta) \qquad \text{and} \qquad \delta_\beta(\lambda) G(\beta)^* = -\rho_\beta(\lambda) H(\beta),$$

which are both valid for $\beta \in \Omega_0$. This in turn leads to (6.10).

Next, for any point $\omega \in \Omega$ at which $G(\omega)$ is invertible, $W_0(\lambda)$ can be reexpressed in the form

$$W_0(\lambda) = W_0(\omega) + \delta_\omega(\lambda) V G(\lambda)^{-1} G(\beta) G(\omega)^{-1} U_0$$

by (2.13). Thus

$$\begin{aligned} W(\lambda) &= W_0(\lambda) D_0 \\ &= D + \delta_\omega(\lambda) V G(\lambda)^{-1} U \end{aligned}$$

with

$$D = W_0(\omega)D_0$$

and

$$U = G(\beta)G(\omega)^{-1}U_0 D_0,$$

which coincide with (6.3) and (6.4), respectively. With the help of the formula

$$H(\omega)PG(\beta) - H(\beta)PG(\omega) = \delta_\beta(\omega)V^*\Sigma_p V, \tag{6.11}$$

it is now readily seen that (6.5) holds. Finally, the verification of (6.6) is an easy consequence of (6.10). ■

In the square case with $\Sigma_p = \Sigma_q = J$, the statement of the last theorem simplifies and, in conjunction with Theorem 5.6, yields the following:

COROLLARY. *Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^{p\times n}$, and $J \in \mathbb{C}^{p\times p}$ be a given set of matrices such that the triple $\{V, A, B\}$ is (a, b) observable, $G(\beta) = a(\beta)A - b(\beta)B$ is invertible for some point $\beta \in \Omega_0$, and J is a signature matrix. Then the following two statements are equivalent:*

(1) *There exists an invertible Hermitian solution P of the equation*

$$A^*PA - B^*PB = V^*JV.$$

(2) *There exists a unique choice of $U_0 \in \mathbb{C}^{n\times p}$ such that the realization*

$$W(\lambda) = \{I_p + \delta_\beta(\lambda)VG(\lambda)^{-1}U_0\}D_0$$

is minimal and J unitary for every choice of $\lambda \in \Omega_0$ at which the indicated inverse exists and every choice of $D_0 \in \mathbb{C}^{p\times p}$ that is J unitary.

Proof. The presumed invertibility of $G(\beta)$ for $\beta \in \Omega_0$ implies that $|a(\beta)| = |b(\beta)| \neq 0$ and hence that $H(\beta)$ is invertible by (5.4), since $\beta' = \beta$. Thus if (1) holds, then Theorem 6.1 is applicable, and formula (6.9) yields the asserted J unitary for the realization $W(\lambda)$ with $U_0 = P^{-1}H(\beta)^{-1}V^*J$. This implies in turn that $\mathcal{R}_c(W) = \mathcal{K}_c(W)$. Moreover, in view of the presumed observability, formula (6.9) also implies that

$$\mathcal{K}_c(W) = \{VG(\lambda)^{-1}x : x \in \mathbb{C}^n\}$$

and is an n-dimensional space. Therefore, by Lemma 3.1,

$$\deg_\rho W = \dim \mathcal{R}_c(W) = n,$$

and thus since A and B belong to $\mathbb{C}^{n\times n}$, the exhibited realization for $W(\lambda)$ is minimal.

The choice of U_0 is unique, because if there was a second choice, say U_1, for which the realization

$$W(\lambda) = \{I_p + \delta_\beta(\lambda) V G(\lambda)^{-1} U_1\} D_0$$

is minimal, then clearly

$$0 = VG(\lambda)^{-1}(U_0 - U_1)W(\beta).$$

Therefore, because of the already established minimality (in fact (a,b) observability would be enough) and the fact that $W(\beta)$ is invertible, $U_0 = U_1$. This completes the proof that (1) $\Rightarrow$ (2).

On the other hand, if (2) holds, then (1) is immediate from Theorem 5.4.

■

We remark that the formulation of this corollary was modeled on the formulation of Theorem 3.5 of [13]. It includes the latter as a special case.

THEOREM 6.2. *Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times n}$, $U \in \mathbb{C}^{n\times q}$, $\Sigma_p \in \mathbb{C}^{p\times p}$, and $\Sigma_q \in \mathbb{C}^{q\times q}$ be a given set of matrices such that:*

(1) *Σ_p and Σ_q are signature matrices with $q \geq p$ and $\mu_\pm(\Sigma_q) \geq \mu_\pm(\Sigma_p)$.*
(2) *There exists an invertible Hermitian matrix $Q \in \mathbb{C}^{n\times n}$ such that*

$$AQA^* - BQB^* = U\Sigma_q U^*. \tag{6.12}$$

(3) *There exists a point $\lambda \in \Omega$ such that $G(\lambda) = a(\lambda)A - b(\lambda)B$ is invertible.*

Then there exist matrices $D_1 \in \mathbb{C}^{p\times q}$ such that

$$D_1 \Sigma_q D_1^* = \Sigma_p \tag{6.13}$$

and points $\beta \in \Omega_0$ such that $G(\beta)$ is invertible. Moreover, for any such choice of D_1 and β and for every point $\omega \in \Omega$ at which $G(\omega)$ is invertible, the matrices

$$D = D_1\{I_q - \rho_\beta(\omega)\Sigma_q U^* G(\beta)^{-*} Q^{-1} G(\omega)^{-1} U\} \tag{6.14}$$

and

$$V = D_1\Sigma_q U^* H(\beta)^{-1}Q^{-1}G(\omega)^{-1}G(\beta) \tag{6.15}$$

are solutions of the equations

$$\Sigma_p = D\Sigma_q D^* + \rho_\omega(\omega)VQV^* \tag{6.16}$$

and

$$D\Sigma_q U^* = VQH(\omega) \tag{6.17}$$

(*which fill out* (5.17)). *The corresponding matrix-valued function*

$$W(\lambda) = D + \delta_\omega(\lambda)VG(\lambda)^{-1}U$$

can be expressed in the form

$$W(\lambda) = D_1 W_1(\lambda),$$

where

$$\begin{aligned} W_1(\lambda) &= I_q + \delta_\beta(\lambda)V_1G(\lambda)^{-1}U \\ &= I_q - \rho_\beta(\lambda)\Sigma_q U^*G(\beta)^{-*}Q^{-1}G(\lambda)^{-1}U \\ &= I_q - \rho_\beta(\lambda)V_1G(\lambda)^{-1}\{G(\beta)QG(\beta)^*\}G(\beta)^{-*}V_1^*\Sigma_q \end{aligned} \tag{6.18}$$

$$V_1 = \Sigma_q U^* H(\beta)^{-1}Q^{-1}, \tag{6.19}$$

$$\frac{\Sigma_q - W_1(\omega)^*\Sigma_q W_1(\lambda)}{\rho_\omega(\lambda)} = U^*G(\omega)^{-*}Q^{-1}G(\lambda)^{-1}U \tag{6.20}$$

and

$$\frac{\Sigma_q - W_1(\lambda)\Sigma_q W_1(\omega)^*}{\rho_\omega(\lambda)} = V_1G(\lambda)^{-1}\{G(\beta)QG(\beta)^*\}G(\omega)^{-*}V_1^*. \tag{6.21}$$

Proof. The existence of a matrix $D_1 \in \mathbb{C}^{p\times q}$, which satisfies (6.13), and a point $\beta \in \Omega_0$ at which $G(\beta)$ (and hence also $H(\beta)$) is invertible is self-evident from the given assumptions. Then clearly the choices $D = D_1$ and $V = D_1V_1$ with V_1 as in (6.19) satisfy (6.16) and (6.17) for $\omega = \beta$. Therefore, by Lemma 5.4,

$$W(\lambda) = D_1\{I_q + \delta_\beta(\lambda)V_1G(\lambda)^{-1}U\}$$

satisfies (5.19), $W\Sigma_q W^{\#} = \Sigma_p$, whereas

$$\begin{aligned} W_1(\lambda) &= I_q + \delta_\beta(\lambda)V_1 G(\lambda)^{-1}U \\ &= I_q - \rho_\beta(\lambda)\Sigma_q U^* G(\beta)^{-*}Q^{-1}G(\lambda)^{-1}U \end{aligned}$$

satisfies (6.20). The last claim is easily checked either by invoking (5.9) and Theorem 5.6 with $\Sigma_p = \Sigma_q = J$ or by direct calculation with the aid of (6.12) and the identity

$$\begin{aligned} &\rho_\beta(\omega)^* G(\lambda)QG(\beta)^* + \rho_\beta(\lambda)G(\beta)QG(\omega)^* \\ &\quad - \rho_\beta(\omega)^*\rho_\beta(\lambda)(AQA^* - BQB^*) \\ &= \rho_\omega(\lambda)G(\beta)QG(\beta)^*, \end{aligned} \tag{6.22}$$

which is valid for every choice of λ, ω in Ω, and $\beta \in \Omega_0$.

The third formula for $W_1(\lambda)$ in (6.18) is easily derived from the first much as in the corresponding step in the proof of Theorem 6.1. This in turn leads to (6.21).

Next, for any point $\omega \in \Omega$ at which $G(\omega)$ is invertible, $W_1(\omega)$ can be reexpressed in the form

$$W_1(\lambda) = W_1(\omega) + \delta_\omega(\lambda)V_1 G(\omega)^{-1}G(\beta)G(\lambda)^{-1}U.$$

Thus

$$\begin{aligned} W(\lambda) &= D_1 W_1(\lambda) \\ &= D + \delta_\omega(\lambda)VG(\lambda)^{-1}U, \end{aligned}$$

with

$$D = D_1 W_1(\omega)$$

and

$$V = D_1 V_1 G(\omega)^{-1}G(\beta),$$

which coincide with (6.14) and (6.15), respectively. With the help of the formula

$$G(\omega)QH(\beta) + \delta_\beta(\omega)U\Sigma_q U^* = G(\beta)QH(\omega),$$

which is valid for every point $\beta \in \Omega_0$, it is now readily checked that (6.17) holds.

Finally, the validation of (6.16) emerges easily from (6.21). ■

In the square case with $\Sigma_p = \Sigma_q = J$, the statement of Theorem 6.2 simplifies and, in conjunction with Theorem 5.6, yields the following counterpart to the preceding corollary:

COROLLARY. *Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times n}$, $U \in \mathbb{C}^{n\times p}$, and $J \in \mathbb{C}^{p\times p}$ be a given set of matrices such that the triple $\{A, B, U\}$ is (a, b) controllable, $G(\beta) = a(\beta)A - b(\beta)B$ is invertible for some point $\beta \in \Omega_0$, and J is a signature matrix. Then the following two statements are equivalent:*

(1) *There exists an invertible Hermitian solution Q of the equation*

$$AQA^* - BQB^* = UJU^*.$$

(2) *There exists a unique choice of $V_1 \in \mathbb{C}^{p\times n}$ such that the realization*

$$W(\lambda) = D_1\{I_p + \delta_\beta(\lambda)V_1G(\lambda)^{-1}U\}$$

is minimal and J unitary for every choice of $\lambda \in \Omega_0$ at which the indicated inverse exists and every choice of $D_1 \in \mathbb{C}^{p\times p}$, which is J unitary.

We omit the proof, since it is easily adapted from the proof of the corollary to Theorem 6.1, except to note that in the proof of (1) $\Longrightarrow$ (2), V_1 is given by formula (6.19) with $\Sigma_q = J$.

The analysis in this section can be extended to cover the nonsymmetric setting, which is discussed in Section 4.3 and Theorem 5.3. Thus (4.8) can be imbedded in (4.21) and (4.24) can be imbedded in (4.25). Some results in this direction have been obtained by Kailath and Sayed [15] for the case $a(\lambda) = 1$ and $b(\lambda) = \lambda$; for additional discussion also see [7], especially the last few pages.

7. FACTORIZATION

In this section we establish factorization formulas for the pair spaces that were considered in Section 4. Although these formulas are derived by purely algebraic manipulations, the pair space structure introduced earlier permits one to establish a one to one correspondence between nondegenerate invariant subspaces and nontrivial factorizations. In a sense these formulas are a natural outgrowth of the factorization formulas that arise in the more familiar setting of matrix-valued functions that are J unitary on Ω_0. To clarify this point of view let us suppose first that W is a matrix-valued function that satisfies assumptions (A1)–(A4) of Section 5 and that W is J unitary on $\Omega_0 \cap \Omega_W$. Then W is said to admit a minimal J unitary factorization if it can be expressed in the form $W(\lambda) = W_1(\lambda)W_2(\lambda)$ for all points λ in the common domain of analyticity in Ω, where:

(1) $W_1(\lambda)$ and $W_2(\lambda)$ also satisfy assumptions (A1)–(A4), but with $1 \leq \deg_\rho W_i \leq n-1$.
(2) $\deg_\rho W_1 + \deg_\rho W_2 = n = \deg_\rho W$.
(3) W_i is J unitary on $\Omega_0 \cap \Omega_{W_i}$ for $i = 1, 2$.

Let us assume further that α has been chosen so that $W(\alpha)$ and $G(\alpha')$ are both invertible in order to ensure that Theorem 5.4 is applicable. These two conditions are automatically met for every point $\alpha \in \Omega_0 \cap \Omega_W$ for which $|a(\alpha)| = |b(\alpha)| \neq 0$ (by the presumed J unitarity of W and Theorem 3.6). The basic underlying principle may then be summarized as follows:

> If $a(\alpha) \neq 0$ [resp. $b(\alpha) \neq 0$], then there is a one to one correspondence between $r(a, b; \alpha)$ [resp. $r(b, a; \alpha)$] invariant subspaces $\mathcal{M}_1$ of $\mathcal{R}_c(W)$, which are nondegenerate with respect to the indefinite inner product $[VG(\lambda)^{-1}u, VG(\lambda)^{-1}v] = v^*Pu$ based on the unique invertible Hermitian solution P of (5.22), and nontrivial J unitary factors W_1 of W: $\mathcal{M}_1 = \mathcal{R}_c(W_1)$, and $W_2 = WW_1^{-1}$.

An analogous formulation for the classical choices of $\rho_\omega(\lambda)$ is given in Theorem 3.2 of [3]; a generalization of the latter to pair spaces is Theorem 4.3 in [4]. It is perhaps good to bear in mind that in the statement of Theorem 3.2 of [3], every space $\mathcal{K}(U_1)$ that is included isometrically in $\mathcal{K}(U)$ is a nondegenerate resolvent invariant subspace of $\mathcal{K}(U)$ and vice versa; for more information see Theorem 3.1 of [3], and Theorem 8.2 of [2] for the groundwork. Explicit formulas for the factors of W are furnished in Theorem 4.2 of [6] when $G(\lambda)$ is of appropriate block upper triangular form. The next step is to identify the invariance alluded to above with the block upper triangular structure of $G(\lambda)$ that is exploited in [6]. We proceed much as in the proof of Lemma 8.1 of [10], but first a normalization.

LEMMA 7.1. *Let $\mathcal{M}_1$ be any k-dimensional subspace of $\mathcal{R}_c(W)$. Then the minimal realization* (2.8) *of $W(\lambda)$ may be chosen in such a way that:*

(1) *$\mathcal{M}_1$ = the span of the first k columns of $F(\lambda) = VG(\lambda)^{-1}$.*
(2) *$G(\alpha) = I_n$.*

Proof. By the presumed minimality, the columns of $F(\lambda)$ form a basis for $\mathcal{R}_c(W)$. Therefore there exists a matrix $Y_1 \in \mathbb{C}^{n\times k}$ of rank k such that

$$\mathcal{M}_1 = \text{the span of the columns of } F(\lambda)Y_1.$$

Let $Y_2 \in \mathbb{C}^{n\times(n-k)}$ be such that

$$Y = [Y_1 \quad Y_2]$$

is invertible and let $X = G(\alpha)^{-1}Y$, $\widetilde{V} = VX$, $\widetilde{G}(\lambda) = Y^{-1}G(\lambda)X$, and $\widetilde{U} = Y^{-1}U$. Then clearly

$$\widetilde{V}\widetilde{G}(\lambda)^{-1}\widetilde{U} = VG(\lambda)^{-1}U,$$

$$\mathcal{M}_1 = \text{the span of the first } k \text{ columns of } \widetilde{V}\widetilde{G}(\lambda)^{-1},$$

and

$$\widetilde{G}(\alpha) = I_n,$$

as needed. ∎

Let $\mathcal{M}_1$ denote the span of the first k columns of $F(\lambda)$ and let

$$G_{ij}(\lambda) = a(\lambda)A_{ij} - b(\lambda)B_{ij}, \qquad i,j = 1,2,$$

denote the corresponding block decomposition of $G(\lambda)$ (i.e., $G_{11}(\lambda)$ is $k\times k$, etc.).

THEOREM 7.1. *If either*

(I) $a(\alpha) \neq 0$ *and* $\mathcal{M}_1$ *is* $r(a,b;\alpha)$ *invariant or,*
(II) $b(\alpha) \neq 0$ *and* $\mathcal{M}_1$ *is* $r(b,a;\alpha)$ *invariant, then the following are equivalent:*

(1) $G_{21}(\alpha) = 0$
(2) $A_{21} = B_{21} = 0$
(3) $G_{21}(\lambda) = 0$ *for every point* $\lambda \in \Omega$.

Conversely, if $A_{21} = B_{21} = 0$, *then* $\mathcal{M}_1$ *is both* $r(a,b;\beta)$ *invariant and* $r(b,a;\beta)$ *invariant for every point* $\beta \in \Omega$ *at which* $G(\beta)$ *is invertible.*

Proof. Let

$$\Pi = \begin{bmatrix} I_k \\ 0 \end{bmatrix}$$

and suppose first that the general assumption (I) is met. Then there exists a $k \times k$ matrix B_1 such that

$$\{r(a,b;\alpha)F\Pi\}(\lambda) = F(\lambda)\Pi B_1.$$

Therefore, by (3.9),

$$F(\lambda)BG(\alpha)^{-1}\Pi = F(\lambda)\Pi B_1$$

and hence, since the columns of $F(\lambda)$ are linearly independent,

$$BG(\alpha)^{-1}\Pi = \Pi B_1.$$

Now if (1) is in force, then this is readily seen to imply that

$$\begin{bmatrix} B_{11}G_{11}(\alpha)^{-1} \\ B_{21}G_{11}(\alpha)^{-1} \end{bmatrix} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

and hence that $B_{21} = 0$. Since $a(\alpha) \neq 0$ and $G_{21}(\alpha) = 0$ by assumption, it follows that $A_{21} = 0$ also. Thus (1) $\Rightarrow$ (2). Since clearly (2) $\Rightarrow$ (3) $\Rightarrow$ (1), this completes the proof of the asserted equivalence under the general assumption (I). The proof under the general assumption (II) goes through in much the same way. Therefore, since the converse statement is self-evident, the proof of the theorem may be deemed complete. ∎

It is perhaps useful to emphasize that in the preceding theorem the normalization $G_{21}(\alpha) = 0$ is automatically met if $G(\alpha) = I_n$.

We turn next to the factorization of a general matrix-valued function $W(\lambda)$ ($= W_{\mathrm{L}}(\lambda)$, which meets the assumptions (A1)–(A4) and for which $W(\alpha)$ and $G(\alpha)$ are both invertible) and its right pairing $W_{\mathrm{R}}(\lambda)$.

THEOREM 7.2. *Let P be an $n \times n$ invertible solution of the matrix equation*

$$A_{\mathrm{R}}^* P A_{\mathrm{L}} - B_{\mathrm{R}}^* P B_{\mathrm{L}} = V_{\mathrm{R}}^* J V_{\mathrm{L}} \tag{7.1}$$

and suppose that the $k \times k$ upper left-hand corner P_{11} of

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is also invertible and that the matrices A_{R}, A_{L}, B_{R}, and B_{L} which intervene in (7.1), *are upper block triangular with respect to the same block decomposition. Let μ be a point in Ω with reflection μ' in Ω such that*

$$a(\mu)a(\mu')^* = b(\mu)b(\mu')^* \neq 0 \tag{7.2}$$

and both

$$G_{\mathrm{L}}(\mu) = a(\mu)A_{\mathrm{L}} - b(\mu)B_{\mathrm{L}} \quad \text{and} \quad G_{\mathrm{R}}(\mu') = a(\mu')A_{\mathrm{R}} - b(\mu')B_{\mathrm{R}} \tag{7.3}$$

are invertible. Then the matrix-valued functions $W_{\rm L}(\lambda)$ *and* $W_{\rm R}(\lambda)$, *which are given by formulas* (4.9) *and* (4.10), *respectively, admit the factorizations*

$$W_{\rm L}(\lambda) = W_{\rm L1}(\lambda)W_{\rm L2}(\lambda) \tag{7.4}$$

and

$$W_{\rm R}(\lambda) = W_{\rm R1}(\lambda)W_{\rm R2}(\lambda), \tag{7.5}$$

with

$$\begin{aligned} W_{\rm L1}(\lambda) &= I - \rho_{\mu'}(\lambda)F_{\rm L1}(\lambda)P_{11}^{-1}F_{\rm R1}(\mu')^*J, \\ W_{\rm L2}(\lambda) &= I - \rho_{\mu'}(\lambda)M_{\rm L}G_{\rm L22}(\lambda)^{-1}Q^{-1}G_{\rm R22}(\mu')^{-*}M_{\rm R}^*J, \\ W_{\rm R1}(\lambda) &= I - \rho_{\mu}(\lambda)F_{\rm R1}(\lambda)P_{11}^{-*}F_{\rm L1}(\mu)^*J, \\ W_{\rm R2}(\lambda) &= I - \rho_{\mu}(\lambda)M_{\rm R}G_{\rm R22}(\lambda)^{-1}Q^{-*}G_{\rm L22}(\mu)^{-*}M_{\rm L}^*J, \end{aligned} \tag{7.6}$$

where

$$\begin{aligned} V_{\rm L} &= [V_{\rm L1} \quad V_{\rm L2}], \qquad V_{\rm R} = [V_{\rm R1} \quad V_{\rm R2}], \\ F_{\rm L}(\lambda) &= [F_{\rm L1}(\lambda) \quad F_{\rm L2}(\lambda)] = [V_{\rm L1} \quad V_{\rm L2}]G_{\rm L}(\lambda)^{-1}, \\ F_{\rm R}(\lambda) &= [F_{\rm R1}(\lambda) \quad F_{\rm R2}(\lambda)] = [V_{\rm R1} \quad V_{\rm R2}]G_{\rm R}(\lambda)^{-1}, \\ F_{\rm L1}(\lambda) &= V_{\rm L1} \quad G_{\rm L11}(\lambda)^{-1}, \qquad F_{\rm R1}(\lambda) = V_{\rm R1}G_{\rm R11}(\lambda)^{-1}, \\ F_{\rm L2}(\lambda) &= V_{\rm L2}G_{\rm L22}(\lambda)^{-1} - V_{\rm L1}G_{\rm L11}(\lambda)^{-1}G_{\rm L12}(\lambda)G_{\rm L22}(\lambda)^{-1}, \\ F_{\rm R2}(\lambda) &= V_{\rm R2}G_{\rm R22}(\lambda)^{-1} - V_{\rm R1}G_{\rm R11}(\lambda)^{-1}G_{\rm R12}(\lambda)G_{\rm R22}(\lambda)^{-1}, \\ \Pi_{\rm L} &= \begin{bmatrix} -P_{11}^{-1}P_{12} \\ I \end{bmatrix}, \qquad \Pi_{\rm R}^* = [-P_{21}P_{11}^{-1} \quad I], \\ M_{\rm L} &= F_{\rm L}(\mu)\Pi_{\rm L}G_{L22}(\mu), \qquad M_{\rm R} = F_{\rm R}(\mu')\Pi_{\rm R}G_{\rm R22}(\mu'), \end{aligned}$$

and

$$Q = P_{22} - P_{21}P_{11}^{-1}P_{12}.$$

If the realization for W *is minimal, then so are the exhibited realizations for* $W_{\rm L1}$, $W_{\rm L2}$, $W_{\rm R1}$ *and* $W_{\rm R2}$ *and the factorizations* (7.4) *and* (7.5).

Proof. The proof is much the same as the proof of Theorem 4.2 in [6] except that extra care must be taken because of the lack of symmetry in the formulas. We give the details because in addition to being more general, the present reorganization is a bit more transparent.

Step 1. The function

$$W_{\rm L}(\lambda) = I - \rho_{\mu'}F_{\rm L}(\lambda)P^{-1}F_{\rm R}(\mu')^*J \tag{7.7}$$

admits a factorization of the form

$$W_{\rm L}(\lambda) = W_{\rm L1}(\lambda)\{I - \rho_{\mu'}(\lambda)W_{\rm L1}(\lambda)^{-1}F_{\rm L}(\lambda)\Pi_{\rm L}Q^{-1}\Pi_{\rm R}^{*}F_{\rm R}(\mu')^{*}J\}. \quad (7.8)$$

Proof of Step 1. Since P and P_{11} are invertible, it follows readily from the well-known factorization formula for P based on the Schur complement that

$$P^{-1} = \begin{bmatrix} I & -P_{11}^{-1}P_{12} \\ 0 & I \end{bmatrix}\begin{bmatrix} P_{11}^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}\begin{bmatrix} I & 0 \\ -P_{21}P_{11}^{-1} & I \end{bmatrix}$$

and hence that

$$\begin{aligned} W_{\rm L}(\lambda) &= I - \rho_{\mu'}(\lambda)[F_{\rm L1}(\lambda) \quad F_{\rm L}(\lambda)\Pi_{\rm L}]\begin{bmatrix} P_{11}^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}\begin{bmatrix} F_{\rm R1}(\mu')^{*} \\ \Pi_{\rm R}^{*}F_{\rm R}(\mu')^{*} \end{bmatrix}J \\ &= W_{\rm L1}(\lambda) - \rho_{\mu'}(\lambda)F_{\rm L}(\lambda)\Pi_{\rm L}Q^{-1}\Pi_{\rm R}^{*}F_{\rm R}(\mu')^{*}J. \end{aligned}$$

But this is clearly equivalent to (7.8).

Step 2 verifies the identity

$$\rho_{\mu'}(\lambda)F_{\rm R1}(\mu')^{*}JF_{\rm L}(\mu)\Pi_{\rm L} = [P_{11} \quad P_{12}]G_{\rm L}(\lambda)G_{\rm L}(\mu)^{-1}\Pi_{\rm L}.$$

Proof of Step 2. Since P is a solution of (7.1), it is readily seen that $[P_{11} \quad P_{12}]$ is a solution of the equation

$$A_{\rm R11}^{*}[P_{11} \quad P_{12}]A_{\rm L} - B_{\rm R11}^{*}[P_{11} \quad P_{12}]B_{\rm L} = V_{\rm R1}^{*}JV_{\rm L}.$$

Consequently, the left-hand side of the asserted identity is equal to

$$\rho_{\mu'}(\lambda)G_{\rm R11}(\mu')^{-*}V_{\rm R1}^{*}JV_{\rm L}G_{\rm L}(\mu)^{-1}\Pi_{\rm L} = \rho_{\mu'}(\lambda)G_{\rm R11}(\mu')^{-*} \times ①,$$

where

$$① = \{A_{\rm R11}^{*}[P_{11} \quad P_{12}]A_{\rm L} - B_{\rm R11}^{*}[P_{11} \quad P_{12}]B_{\rm L}\}\, G_{\rm L}(\mu)^{-1}\Pi_{\rm L}.$$

But now as

$$[P_{11} \quad P_{12}]a(\mu)A_{\rm L}G_{\rm L}(\mu)^{-1}\Pi_{\rm L} = [P_{11} \quad P_{12}]b(\mu)B_{\rm L}G_{\rm L}(\mu)^{-1}\Pi_{\rm L}$$

and (7.2) is in force, it follows that the factor ① can be expressed in two different ways:

$$\begin{aligned} ① &= \left\{A_{\rm R11}^{*} - \frac{a(\mu)}{b(\mu)}B_{\rm R11}^{*}\right\}[P_{11} \quad P_{12}]A_{\rm L}G_{\rm L}(\mu)^{-1}\Pi_{\rm L} \\ &= \frac{G_{\rm R11}(\mu')^{*}}{a(\mu')^{*}}[P_{11} \quad P_{12}]A_{\rm L}G_{\rm L}(\mu)^{-1}\Pi_{\rm L}, \end{aligned}$$

and, upon "eliminating A_{L} instead of B_{L},"

$$\begin{aligned}
① &= \left\{\frac{b(\mu)}{a(\mu)}A^*_{\mathrm{R}11} - B^*_{\mathrm{R}11}\right\}[P_{11} \quad P_{12}]B_{\mathrm{L}}G_{\mathrm{L}}(\mu)^{-1}\Pi_{\mathrm{L}} \\
&= \frac{G_{\mathrm{R}11}(\mu')^*}{b(\mu')^*}[P_{11} \quad P_{12}]B_{\mathrm{L}}G_{\mathrm{L}}(\mu)^{-1}\Pi_{\mathrm{L}}.
\end{aligned}$$

Thus

$$\rho_{\mu'}(\lambda)G_{\mathrm{R}11}(\mu')^{-*} \times ① = [P_{11} \quad P_{12}]G_{\mathrm{L}}(\lambda)G_{\mathrm{L}}(\mu)^{-1}\Pi_{\mathrm{L}},$$

as needed.

Step 3 verifies the identity

$$W_{\mathrm{L}1}(\lambda)F_{\mathrm{L}}(\mu)\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\mu) = F_{\mathrm{L}}(\lambda)\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\lambda). \tag{7.9}$$

Proof of Step 3. In view of (7.6) the left-hand side of (7.9) is equal to

$$F_{\mathrm{L}}(\mu)\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\mu) - \rho_{\mu'}(\lambda)F_{\mathrm{L}1}(\lambda)P_{11}^{-1}F_{\mathrm{R}1}(\mu')^*JF_{\mathrm{L}}(\mu)\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\mu).$$

Therefore by Step 2, the asserted identity is equivalent to showing that

$$\begin{aligned}
&F_{\mathrm{L}}(\mu)\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\mu) - F_{\mathrm{L}}(\lambda)\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\lambda) \\
&\quad = F_{\mathrm{L}1}(\lambda)P_{11}^{-1}[P_{11} \quad P_{12}]G_{\mathrm{L}}(\lambda)G_{\mathrm{L}}(\mu)^{-1}\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\mu).
\end{aligned} \tag{7.10}$$

The left-hand side of (7.10) is equal to

$$② = [V_{\mathrm{L}1} \quad 0]\left\{G_{\mathrm{L}}(\mu)^{-1}\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\mu) - G_{\mathrm{L}}(\lambda)^{-1}\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\lambda)\right\},$$

since

$$[0 \quad V_{\mathrm{L}2}]\left\{G_{\mathrm{L}}(\mu)^{-1}\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\mu) - G_{\mathrm{L}}(\lambda)^{-1}\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\lambda)\right\} = 0.$$

Furthermore, upon setting

$$③ = [V_{\mathrm{L}1} \quad 0]G_{\mathrm{L}}(\mu)^{-1}\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\mu),$$

it is readily seen that

$$\begin{aligned}
② &= ③ - [V_{\mathrm{L}1} \quad 0]G_{\mathrm{L}}(\lambda)^{-1}\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\lambda) \\
&= ③ - [F_{\mathrm{L}1}(\lambda) - F_{\mathrm{L}1}(\lambda)G_{\mathrm{L}12}(\lambda)G_{\mathrm{L}22}(\lambda)^{-1}]\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\lambda) \\
&= ③ + F_{\mathrm{L}1}(\lambda)P_{11}^{-1}\{P_{12}G_{\mathrm{L}22}(\lambda) + P_{11}G_{\mathrm{L}12}(\lambda)\}.
\end{aligned}$$

At the same time, the right-hand side of (7.10) is clearly equal to

$$\begin{aligned} &F_{\mathrm{L}1}(\lambda)P_{11}^{-1}[P_{11}G_{L11}(\lambda) \quad P_{11}G_{\mathrm{L}12}(\lambda)+P_{12}G_{\mathrm{L}22}(\lambda)]G_{\mathrm{L}}(\mu)^{-1}\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\mu) \\ &\quad = F_{\mathrm{L}1}(\lambda)\{[G_{\mathrm{L}11}(\lambda) \quad 0]+P_{11}^{-1}[0 \quad P_{11}G_{\mathrm{L}12}(\lambda)+P_{12}G_{\mathrm{L}22}(\lambda)]\} \\ &\qquad \times G_{\mathrm{L}}(\mu)^{-1}\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\mu) \\ &\quad = ③ + F_{\mathrm{L}1}(\lambda)P_{11}^{-1}\{P_{11}G_{\mathrm{L}12}(\lambda)+P_{12}G_{\mathrm{L}22}(\lambda)\}. \end{aligned}$$

This completes the proof of this step.

Step 4 completes the proof of the theorem.

Proof of Step 4. In order to establish (7.4) it remains to show that the term inside the curly brackets in (7.8) is equal to the asserted formula for $W_{\mathrm{L}2}(\lambda)$. But in view of Step 3, that term is equal to

$$\begin{aligned} &I-\rho_{\mu'}(\lambda)F_{\mathrm{L}}(\mu)\Pi_{\mathrm{L}}G_{\mathrm{L}22}(\mu)G_{\mathrm{L}22}(\lambda)^{-1}Q^{-1}\Pi_{\mathrm{R}}^{*}F_{\mathrm{R}}(\mu')^{*}J \\ &\quad = I-\rho_{\mu'}(\lambda)M_{\mathrm{L}}G_{\mathrm{L}22}(\lambda)^{-1}Q^{-1}\Pi_{\mathrm{R}}^{*}F_{\mathrm{R}}(\mu')^{*}J. \end{aligned}$$

This does the trick since

$$\Pi_{\mathrm{R}}^{*}F_{\mathrm{R}}(\mu')^{*} = G_{\mathrm{R}22}(\mu')^{-*}M_{\mathrm{R}}^{*},$$

by the definition of M_{R}.

The proof of all the advertised formulas may now be deemed complete since (7.5) follows from (7.4) by an appropriate change of notation: $\mathrm{L}\to\mathrm{R}$, $\mathrm{R}\to\mathrm{L}$, $\mu\to\mu'$, $\mu'\to\mu$ and $P\to P^{*}$. (In connection with the latter it is good to note that $(P^{*})_{12}=(P_{21})^{*}$, which is written as P_{21}^{*} for short.) In case of doubt it is also possible to derive (7.5) from scratch by mimicking the proof of (7.4). The formulas

$$\rho_{\mu}(\lambda)F_{\mathrm{L}1}(\mu)^{*}JF_{\mathrm{R}}(\mu')\Pi_{\mathrm{R}} = [P_{11}^{*} \quad P_{21}^{*}]G_{\mathrm{R}}(\lambda)G_{\mathrm{R}}(\mu')^{-1}\Pi_{\mathrm{R}}$$

and

$$W_{\mathrm{R}1}(\lambda)F_{\mathrm{R}}(\mu')\Pi_{\mathrm{R}}G_{\mathrm{R}22}(\mu') = F_{\mathrm{R}}(\lambda)\Pi_{\mathrm{R}}G_{\mathrm{R}22}(\lambda)$$

play the role of Steps 2 and 3, respectively.

Finally, the assertions on minimality are clear from the formulas for the factors and the rules for multiplication exhibited in Section 2.3 just as in the classical case. Thus, for example,

$$n = \deg_{\rho}W_{\mathrm{L}}(\lambda) \le \deg_{\rho}W_{\mathrm{L}1}(\lambda)+\deg_{\rho}W_{\mathrm{L}2}(\lambda),$$

whereas

$$\deg_\rho W_{L1}(\lambda) = k \qquad \text{and} \qquad \deg_\rho W_{L2}(\lambda) \leq n - k.$$

Thus the last inequality must in fact be an equality, otherwise we obtain a contradiction of the presumed minimality of the realization for $W_L(\lambda)$. The proof for the factors of $W_R(\lambda)$ goes through in just the same way. ■

THEOREM 7.3. *In the setting of Theorem* 7.1,

$$A_{R22}^* Q A_{L22} - B_{R22}^* Q B_{L22} = M_R^* J M_L.$$

Proof. Upon multiplying (7.1) through by $a(\mu)a(\mu')^* = b(\mu)b(\mu')^*$ it is readily seen that

$$\begin{aligned} a(\mu')^* V_R^* J a(\mu) V_L &= a(\mu')^* A_R^* P a(\mu) A_L - b(\mu')^* B_R^* P b(\mu) B_L \\ &= \{G_R(\mu')^* + b(\mu')^* B_R^*\} P a(\mu) A_L + b(\mu')^* B_R^* P \\ &\quad \times \{G_L(\mu) - a(\mu) A_L\} \\ &= G_R(\mu')^* P a(\mu) A_L + b(\mu')^* B_R^* P G_L(\mu). \end{aligned}$$

Therefore,

$$P a(\mu) A_L G_L(\mu)^{-1} + G_R(\mu')^{-*} b(\mu')^* B_R^* P = a(\mu')^* F_R(\mu')^* J F_L(\mu) a(\mu).$$

But now as

$$\Pi_R^* P = [0 \quad Q] \qquad \text{and} \qquad P \Pi_L = \begin{bmatrix} 0 \\ Q \end{bmatrix},$$

it follows easily, upon multiplying the last formula through by $G_{R22}(\mu')^* \Pi_R^*$ on the left and by $\Pi_L G_{L22}(\mu)$ on the right, that

$$G_{R22}(\mu')^* Q a(\mu) A_{L22} + b(\mu')^* B_{R22}^* Q G_{L22}(\mu) = a(\mu')^* M_R^* J M_L a(\mu).$$

Next, upon substituting the full formulas for $G_{R22}(\mu')$ and $G_{L22}(\mu)$ in the left-hand side of the last line it is easily seen to reduce to

$$a(\mu')^* a(\mu) \{A_{R22}^* Q A_{L22} - B_{R22}^* Q B_{L22}\}.$$

The rest is self-evident. ■

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